Math 311 - Introduction to Proof and Abstract Mathematics
Group Assignment # 21 **Name:**________________________________
Due: At the end of class on Tuesday, April 23rd

Equivalence Classes and Partitions:

Recall – Definition 7.2.1: Let $\sim$ be a relation on a set A.

- We say that $\sim$ is **reflexive** if for all $a \in A$, $a \sim a$.
- We say that $\sim$ is **symmetric** if for all $a, b \in A$, if $a \sim b$ then $b \sim a$.
- We say that $\sim$ is **transitive** if for all $a, b, c \in A$, if $a \sim b$ and $b \sim c$, then $a \sim c$.
- The relation $\sim$ is called an **equivalence relation** on A if $\sim$ is reflexive, symmetric, and transitive.

1. In a previous example, we defined a relation $\sim$ on $\mathbb{Z} \times \mathbb{Z}^*$ as follows:

   For all $a, c, \in \mathbb{Z}$ and all $b, d \in \mathbb{Z}^*$, $(a, b) \sim (c, d)$ if and only if $ad = bc$. Note that the elements of $\sim$ are actually ordered pairs of ordered pairs. That is, $\sim$ consists of elements of the form $((a, b), (c, d))$. For example, $(2, 4) \sim (3, 6)$ since $(2) \cdot (6) = (4) \cdot (3)$.

   (a) Determine whether or not $\sim$ is reflexive. Fully justify your answer.

   (b) Determine whether or not $\sim$ is symmetric. Fully justify your answer.

   (c) Determine whether or not $\sim$ is transitive. Fully justify your answer.

   (d) Based on what you found above, is $\sim$ an equivalence relation? Why or why not?

Definition 7.2.5: Let $\sim$ be an equivalence relation on a nonempty set X and let $a \in X$. The **equivalence class** of a is the set $[a] = \{x \in X \,|\, x \sim a\}$. The set of all equivalence classes of $\sim$ is denoted by $X/\sim = \{[a] \,|\, a \in X\}$. Note that since $X/\sim$ is a collection of subsets of X, $X/\sim \subseteq \mathcal{P}(X)$.

Example 1: Our first (and perhaps most straightforward) example of equivalence classes are the *congruence classes* modulo n that we looked at on Assignment 20. We developed a complete listing of the congruence (equivalence) classes when $n = 5$ on that assignment. Recall that we proved at the end of Assignment 21 that congruence modulo n is an equivalence relation. We can carry out a similar process for any value of n. We also looked at some nice properties of these sets that will end of being true for any set of equivalence classes for an equivalence relation $\sim$.

That is, if $\equiv_m$ is the relation congruence modulo m on the set $\mathbb{Z}$, then for any $a \in \mathbb{Z}$, $[a] = [a]_m = \{b \in \mathbb{Z} \,|\, b \equiv a \bmod m\}$. The set of all equivalence classes is $\mathbb{Z}/\equiv_m = \mathbb{Z}_m = \{[0]_m, [1]_m, \cdots, [m-1]_m\}$. Note that we do not need to list any additional sets, as once we have a set or each "canonical representative", we have a complete listing of all possible equivalence classes.

2. Let $\sim$ be the equivalence relation on $\mathbb{Z}$ defined by $m \sim n$ if and only if $m^2 = n^2$.

   (a) Prove that $\sim$ as defined above is an equivalence relation.

   (b) Find the equivalence classes [3], [17] and [0]. Then give a general description of equivalence classes for this equivalence relation.

3. Recall the relation defined above: for all $a, c, \in \mathbb{Z}$ and all $b, d \in \mathbb{Z}^*$, $(a, b) \sim (c, d)$ if and only if $ad = bc$.

   (a) Find $[(1, 2)]$

   (b) Find $[(4, 3)]$

   (c) Give a general description of the equivalence classes for this relation.

Theorem 7.2.9: Let $\sim$ be an equivalence relation on a nonempty set X.

- For all $a \in X$, $a \in [a]$.
- For all $a, b \in X$, $a \sim b$ if and only if $[a] = [b]$.
- For all $a, b \in X$, $a \not\sim b$ if and only if $[a] \cap [b] = \emptyset$.

Proof:

To prove the first part, let $a \in X$. Since $\sim$ is reflexive, we have $a \sim a$. Hence, by Definition 7.2.5, $a \in [a]$.

To prove the second part, suppose $a, b \in X$.

"$\Rightarrow$" Assume $a \sim b$. We must show that $[a] = [b]$. Since $[a]$ and $[b]$ are sets, we will use a standard set containment argument to show equality. Suppose that $x \in [a]$. Then, by definition of $[a]$, $x \sim a$. We also have that $a \sim b$, so, by transitivity, we must have $x \sim b$. Hence $x \in [b]$, and thus $[a] \subseteq [b]$. Similarly, if we suppose that $x \in [b]$, then $x \sim b$. Since $a \sim b$, since $\sim$ is symmetric, $b \sim a$. Hence, since $x \sim b$ and $b \sim a$, by transitivity, we have $x \sim a$. Therefore $x \in [a]$, and hence $[b] \subset [a]$. Thus $[a] = [b]$.

"$\Leftarrow$" Assume that $[a] = [b]$. We must show that $a \sim b$. By the previous part of this theorem, $a \in [a]$. therefore, since $[a] = [b]$, $a \in [b]$. Thus, by definition of $[b]$, $a \sim b$.

To prove the third part, we will use proof by contraposition.

"$\Rightarrow$" Assume that $[a] \cap [b] \neq \emptyset$. We must show that $a \sim b$. Since $[a] \cap [b] \neq \emptyset$, there is some element $x \in X$ such that $x \in [a] \cap [b]$. Since $x \in [a]$, $x \sim a$. Thus, since $\sim$ is symmetric, $a \sim x$. Similarly, since $x \in [b]$, $x \sim b$. Therefore, since $a \sim x$ and $x \sim b$, since $\sim$ is transitive, $a \sim b$, as desired.

"$\Leftarrow$" Assume that $a \sim b$. We must show that $[a] \cap [b] \neq \emptyset$. As above, $a \in [a]$. Moreover, since $a \sim b$, then $a \in [b]$. Hence $a \in [a] \cap [b]$. This $[a] \cap [b] \neq \emptyset$. $\square$.

Definition 7.3.1 Let X be a nonempty set, and let $\mathbf{P}$ be a collection of subsets of X (i.e. $\mathbf{P} \subseteq \mathcal{P}(X)$). The collection $\mathbf{P}$ is a **partition** of X if:

- For all $A \in \mathbf{P}$, $A \neq \emptyset$.
- For all $A, B \in \mathbf{P}$, $A = B$, or $A \cap B = \emptyset$.
- For all $x \in X$, there exists $A \in \mathbf{P}$ such that $x \in A$ (that is, the sets in $\mathbf{P}$ **cover** X).

Example: Let $X = \{a, b, c, d, e\}$. Then $\mathbf{P} = \{\{b, d\}, \{a\}, \{c, e\}\}$ is a partition of X.

Corollary 7.3.3 (Corollary to Theorem 7.2.9). Let $\sim$ be an equivalence relation on a nonempty set X. Then $X/\sim = \{[a] \mid a \in X\}$, the set of equivalence classes of $\sim$ is a partition of X.

4. Let $X = \{a, b, c, d, e, f, g\}$.

   (a) Find a partition of X involving exactly three subsets of X.

   (b) Find a partition of X involving more than four subsets of X.

   (c) Find a collection of subsets of X whose union is X but that **do not** form a partition of X. Explain why your example fails to be a partition.

Theorem 7.3.5 Let $\mathbf{P}$ be a partition of a nonempty set X. Define a relation $\sim$ on X for all $a, b \in X$ by defining:

$$a \sim b \Leftrightarrow (\exists A \in \mathbb{P})[a \in A \wedge b \in A].$$

Then $\sim$ is an equivalence relation on X. Furthermore, the equivalence classes of $\sim$ are exactly the elements of the partition $\mathbf{P}$; that is, $X/\sim = \mathbf{P}$.

Proof: See page 164 in your textbook.

5. Let $A = \{a, b, c, d, e, f\}$. Give a complete listing of the ordered pairs in the equivalence relation $\sim$ generated by the partition $\mathbf{P} = \{\{a, c, e\}, \{b, f\}, \{d\}\}$