Math 311 - Introduction to Proof and Abstract Mathematics
Group Assignment # 6 **Name:**____________________________
Due: Tuesday, September 18th

Properties of Real Numbers: Let a, b, and c be arbitrary real numbers.

Closure under + and $\cdot$	$a+b$ and ab are also real numbers.
Associative properties	$(a+b)+c = a+(b+c)$ and $(ab)c = a(bc)$.
Commutative properties	$a+b = b+a$ and $ab = ba$.
Distributive property	$a(b+c) = ab+ac$.
Identities	$0 \neq 1$, $a+0 = a$, $a \cdot 1 = a$, and $a \cdot 0 = 0$.
Additive inverses	$\forall a \exists!$ real number $-a = -1 \cdot a$ such that $a + (-a) = 0$.
Subtraction	$b - a$ is defined to equal $b + (-a)$.
Multiplicative Inverses	If $a \neq 0$ then $\exists!$ real number $a^{-1} = \frac{1}{a}$ such that $a\dot{a}^{-1} = a \cdot \frac{1}{a} = 1$.
Division	When $a \neq 0$ $\frac{b}{a}$ is defined to equal $b \cdot \frac{1}{a}$.
Cancellation	If $ab = ac$ and $a \neq 0$, then $b = c$.
Transitive property of $<$	If $a < b$ and $b < c$, then $a < c$.
Trichotomy	Exactly one of $a < b$, $a = b$ or $a > b$ holds.
Order property 1	If $a < b$ then $a + c < b + c$.
Order property 2	If $c > 0$, then $a < b$ iff $ac < bc$.
Order property 3	If $c < 0$, then $a < b$ iff $ac > bc$.

1. Our goal in this problem is to Prove or Disprove the following statement. If x and y be distinct positive real numbers, then $\frac{x}{y} + \frac{y}{x} > 2$. In situations where it is not immediately clear whether the statement is true or false, there are two helpful techniques that we can employ.

   Exploratory Examples: In order to build intuition about whether or not we believe a result, we can look at a few specific examples that satisfy the hypotheses and determine whether or not the desired conclusion holds. If we find several well chosen examples that work, we begin to believe the result and can start examining patterns in the positive examples in order to find ideas that would lead us to create a valid proof. We also might stumble upon a counterexample that serves to disprove the statement.

   (a) Compute $\frac{x}{y} + \frac{y}{x}$ for at least three different pairs of distinct positive values of x and y. Comment on whether you find evidence that the given statement is true or if you find a counterexample to the statement.

   Working Backwards: A second technique that we can use to explore whether or not a statement is true or false (and help us work toward a proof if it ends up being true) is to work backwards. To do this, we temporarily assume that the desired conclusion holds and we explore consequences of that assumption. In practice, this can either lead to a key idea that we can use to prove the result or it can lead to a logical contradiction that demonstrates that the statement is false. Note that **it is not acceptable to assume the result we are proving** in a formal proof, so everything we deduce when working backwards will have to be reworked if we move on to write a formal proof.

   To see how this applies to the statement we are considering, let's begin by assuming that $\frac{x}{y} + \frac{y}{x} > 2$. Since we know that x and y are positive, we can multiply both sides of this inequality by xy without reversing the inequality. That is, $xy\left(\frac{x}{y} + \frac{y}{x}\right) > 2xy$, or $x^2 + y^2 > 2xy$. Then $x^2 - 2xy + y^2 > 0$, which factors to give $(x-y)^2 > 0$. We can now take note that since x and y are distinct, $x - y$ is non-zero (which tells us something about the value of its square...).

(b) Beginning with two distinct positive real numbers x and y, reverse the computations that we carried out above in order to construct a valid proof of the original statement above. Carefully justify each step in your proof.

Definition 2.1.11 Given $x \in \mathbb{R}$, the **absolute value of** x, denoted by $|x|$, is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

2. Prove or Disprove: Let $x, y \in \mathbb{R}$. $|x| = |y|$ if and only if $x = y$.

3. Prove or Disprove: Let $x, y \in \mathbb{R}$. If $x > y$, then $|x| > |y|$.

4. Prove or Disprove: If $a \in \mathbb{R}$ then $|a| = |-a|$.

Uniqueness Proofs: As we previously discussed, the quantifier $\exists!$ means "there exists a unique element" or "exactly one element in the universe" satisfies the statement. In order to prove that such a statement holds, we must do two things:

- We must show that there is *at least one* element in the universe that satisfies the statement. We call this the **Existence** portion of the proof.
- We must show that there **is not** a second element of the universe that satisfies the statement. We generally do this by showing that two potential elements that both satisfy the statement must be equal. We call this the **Uniqueness** portion of the proof.

Example: Prove that there is a unique element $x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$, $x + y = y$.

Proof: (Existence) Let $x = 0$. Notice that, by from the Identity Property of real numbers, $0 + y = y$ for all $y \in \mathbb{R}$.

(Uniqueness) Suppose that $a + y = y$ for all $y \in \mathbb{R}$. Then, in particular, if $y = 0$, then $a + y = y$, so $a + 0 = 0$. However, using commutativity of addition, $a + 0 = 0 + a$, and we know that $0 + a = a$. Hence, by transitivity of equality, $0 = a + 0 = 0 + a = a$. Hence $a = 0$.

5. Prove that 1 is the unique multiplicative identity for the real numbers.

Proof by Contraposition: Recall that when we looked at conditional statements last chapter, we showed that a conditional statement of the form $P \Rightarrow Q$ and its related contrapositive statement $\neg Q \Rightarrow \neg P$ are logically equivalent. With this in mind, if we would like to prove that a conditional statement is true, when it is convenient to do so, we can choose to prove the related contrapositive statement instead. Doing so proves the original conditional statement because they are logically equivalent.

6. Consider the statement: Let $n \in \mathbb{Z}$. If n^2 is even then n is even.

   (a) Write the contrapositive of this statement.

   (b) Write a proof of the contrapositive statement you wrote in part (a).

   (c) Explain, in your own words, why this allows us to conclude that the original statement is true.

Proving Biconditional Statements: Logically speaking, a biconditional statement is two conditional statements combined together into a single statement. With this in mind, to prove a biconditional statement of the form $P \Leftrightarrow Q$, we will generally combine two separate proofs. A proof that $P \Rightarrow Q$ and a proof that $Q \Rightarrow P$.

7. Prove that for any integers n, n is even if and only if n^2 is even (notice that you already proved half of this statement above).

Presentation Problem: Consider the series: $1^1, 2^2, 3^3, 4^4, \cdots, n^n, \cdots$
First, rewrite this as a series of integers: $1, 4, 27, 256, \cdots$
Then, keep only the final digit of each integer: $1, 4, 7, 6, \cdots$
(a) Show that this series is periodic.
(b) Find its repeating part.