Math 311 - Introduction to Proof and Abstract Mathematics
Group Assignment # 7
Due: Thursday, February 7th

Name:________________________________

1. Consider the statement: If x is a real number satisfying $x^2 + 3x - 4 > 0$, then $x < -4$ or $x > 1$.

   (a) Write the contrapositive of the statement given above (in the simplest way possible).

   (b) Use proof by contraposition to prove the original statement.

Proving "Or" Statements:

2. Use truth tables to verify that the statements $P \vee Q$ and $\neg P \Rightarrow Q$ are logically equivalent.

3. Without using truth tables, explain why $P \vee Q$ and $\neg Q \Rightarrow P$ are also logically equivalent.

With this in mind, we will often use the following strategy to prove a statement of the form $P \vee Q$:

To prove that a statement of the form $P \vee Q$ is true
• We can begin by assuming $\neg P$ and prove that Q must be true. • Equivalently, if we prefer, we can begin by assuming $\neg Q$ and prove that P must be true.

4. Prove the following statement: If $|b| > 2$ then $b > 2$ or $b < -2$. [Hint: look at the contrapositive statement]

Recall: The set of **rational numbers** is defined as $\mathbb{Q} = \{x : \text{there exist } a, b \in \mathbb{Z}, b \neq 0 \text{ such that } x = \frac{a}{b}\}$. Any real number that **does not** satisfy the definition of a rational number is called an **irrational number**.

5. Prove or disprove the following "closure" properties for rational numbers.

   (a) The sum of two rational numbers is always a rational number.

   (b) The product of two rational numbers is always a rational number.

   (c) The quotient of two rational numbers is always a rational number.

6. Given an example (without proof) of a number that is irrational.

Proof by Contradiction: Recall that a statement that is never true, such as $Q \wedge \neg Q$ is called a **contradiction**. One way to prove a statement P is to assume the opposite (the negation of P) $\neg P$ holds and to show that this assumption leads to a contradiction. This method of proof is considered to be a "indirect proof" method called **proof by contradiction**. Here is an outline of this proof method.

To prove that a statement P is true by contradiction
• Begin by assuming $\neg P$ is true.
• Use a logical sequence of steps to deduce a contradiction from your initial assumption.
• Conclude that P must be true.

7. In your own words, explain why **Proof by Contradiction** works and makes logical sense.

Proposition 2.2.1: There do not exist integers m and n such that $12m + 9n = 100$.

Proof: In order to obtain a contradiction, suppose that there **are** integers m and n satisfying $12m + 9n = 100$. Then, factoring, we have $3(4m + 3n) = 100$. Since $4m + 3n$ is an integer (by the closure properties of integer arithmetic), we then have that $3|100$. That is, there is some integer k such that $3k = 100$. Note that if $k = 33$, $3k = 99$, and if $k = 34$, $3k = 102$. Moreover, by the order properties of real numbers, if $\ell < 33$, $3\ell < 99$, and if $\ell > 34$, $3\ell > 102$. Hence, there is no integer k such that $3k = 100$. This is a contradiction. Therefore, there do not exist integers m and n such that $12m + 9n = 100$. $\square$.

8. Use proof by contradiction to prove that the sum of a rational number and an irrational number is irrational.