Math 311 - Introduction to Proof and Abstract Mathematics
Group Assignment # 9 **Name:**____
Due: By 4:00pm on Friday, February 15th

Recall: Back in Chapter 1, we defined our two main quantifiers: $\forall$, the "Universal" Quantifier, and $\exists$, the "Existential" Quantifier (we also defined $\exists!$, the "Uniqueness" Quantifier). When a statement contains only one type of quantifier, the order of quantification does not usually matter. When we are given a statement that involves more than one quantifier, we say that the statement has **mixed quantifiers**. In statements involving mixed quantifiers, the order of quantification *does* impact the meaning (and hence the truth value) of the statement. As we parse a statement with multiple quantifiers, we must select appropriate elements in the exact order they are presented in the statement.
Example: Let x and y be integers and consider the following statements involving mixed quantifiers.

- $\forall x[\exists y(x + y = 0)]$
- $\exists y[\forall x(x + y = 0)]$

The first statement is true. To see this, let x be an arbitrary integer. If we set $y = -x$, then y is also an integer and we have $x + y = x + (-x) = 0$. For example, if $x = 4$, we take $y = -4$. Then $x + y = 4 + (-4) = 0$. Note that simply working out one example would not be a proof. Since we have given a general construction that works for any integer x, we have proven that the statement is true. The example merely serves to illustrate the general construction.

The second statement is false. In order for this statement to be true we would need to be able to pick a specific y in such a way that $x + y = 0$ for **all** integers x. To show that this statement is false, suppose such an integer y exists. Notice that $x = 0$ and $x = 1$ are both legal choices for x. Then we must have $0 + y = 0$, so $y = 0$. Similarly, we must have $1 + y = 0$, so $y = -1$. Thus, by the transitivity of equality, $0 = -1$. this is impossible, so our assumption that a suitable y exists must be in error. Hence no such y exists, and thus the original statement is false.

1. Let x, y and z be integers. Determine the truth value of each of the following quantified statements.

   (a) $\forall x[\exists y(x + y = 1)]$

   (b) $\forall x[\forall y[\exists z(xy < z)]]$

   (c) $\exists z[\forall x[\forall y(xy < z)]]$

Definition 2.4.2: Let f be a function of a single variable defined for all real numbers in an open interval containing the real number a, except possibly at a itself, and let L be a real number. Then the **limit** of f at a is L, denoted $\lim_{x \to a} f(x) = L$ if $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x)[0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon]$.

To make sense of this, note that the idea behind this formal definition is that in order for the value of $f(x)$ to approach L as the input x approaches a, we must be able to make the value of $f(x)$ arbitrarily close to (within ϵ of) L by selecting inputs within δ of the x-value a.

2. Draw a diagram that illustrates definition 2.4.2.

3. Let $f(x) = 4x - 3$. Prove that $\lim_{x \to 3} f(x) = 9$ [I will help get you started.]

   Proof: Let $\epsilon > 0$ be given. We must find a real number $\delta > 0$ such that $(\forall x)[0 < |x - 3| < \delta \Rightarrow |f(x) - 9| < \epsilon$

   (a) Let $\epsilon = 0.1$ and find a corresponding δ that satisfies the required inequality.

   (b) Let $\epsilon = 0.01$ and find a corresponding δ that satisfies the required inequality.

   (c) Complete a full proof by using the ideas from these two specific examples to define δ in terms of ϵ for a general ϵ value and "working backwards" to produce a convincing formal argument.

4. The remainder of this handout consists of general presentation problems that do not need to be handed in for grading but that you can present on the board during class today.

   (a) Prove that if n is an integer and that $n^2 + 11$ is even, then n is odd.

   (b) Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$

   (c) Prove that given any two rational numbers $p < q$, there is a rational number r with $p < r < q$.

   (d) Prove or disprove: Every non-negative integer can be written as the sum of at most 3 perfect squares.

   (e) Prove or disprove: Let a, b, and c be integers. If $a|b$ and $a|c$, then $a|(b + c)$.

   (f) Prove or disprove: Let a, b, c and d be integers. If $a|b$ and $c|d$, then $ac|bd$.

   (g) Prove that $x^2 + y^2 = 11$ has no integer solutions.

   (h) Prove or disprove: If a and b are positive real numbers, then $a + b \geq 2\sqrt{ab}$.

   (i) Prove or disprove: If a does not divide bc, then a does not divide b.

   (j) Formulate a conjecture about the decimal digits that appear as the final digit of the fourth power of an integer. Prove your conjecture using proof by cases.