Math 311
Presentation Problems

Part 1: The following is a list of presentation problems form earlier in the semester that are still eligible to be presented.

1. Prove or Disprove: if $A - C \subseteq B - C$, then $A \subseteq B$.

2. Use mathematical induction to prove that $x^2 - 1$ is divisible by 8 whenever x is a *positive odd integer.*

3. Formulate a conjecture about the decimal digits that appear as the final digit of the fourth power of an integer. Prove your conjecture using proof by cases.

4. Express the set $\{1\}$ as the intersection of a collection of distinct, non-empty intervals in $\mathbb{R}$ indexed by $\mathbb{Z}^+$.

5. Use the properties of Real Numbers to prove that given $a, b \in \mathbb{R}$, $ab = 0$ if and only if $a = 0$ or $b = 0$.

6. Let $\mathcal{P}$ be the set of all polynomial functions. Let $D : \mathcal{P} \to \mathcal{P}$ be the function given by $D(p(x)) = p'(x)$. That, the function that maps each function to its derivative.

   (a) Find $im\, D$.

   (b) Let $\mathcal{L} = \{p(x) \in \mathcal{P} \,|\, p(x) = ax + b$. Find the preimage of $\mathcal{L}$ under D.

   (c) If D 1-1? Justify your answer.

7. Let $h(x) = \sqrt{x^2 + 1} - 5$. Find three different ways to express $h(x)$ as the composition of two functions (that is, find three distinct pairs of functions f and g so that $h = g \circ f$.

8. Prove Corollary 5.4.10

9. Let $a_1, a_2, b_1, b_2, c \in \mathbb{Z}$. and assume that $a_1 \equiv a_2 \bmod m$ and $b_1 \equiv b_2 \bmod m$. Prove that $a_1 + b_1 \equiv a_2 + b_2 \bmod m$.

10. For each of the following relations (each relation is considered a separate presentation problem):

    - Determine whether or not the relation is an equivalence relation. If the relation is an equivalence relation, describe its equivalence classes.
    - Determine whether or not the relation is a poset. For those that are, either find a pair of incomparable elements or explain why this is not possible.

    (a) The relation $\sim$ on $\mathbb{R}$ defined by $a \sim b$ if and only if $|a| = |b|$.

    (b) The relation $\sim$ on the set of students as MSUM defined by $a \sim b$ if and only if person a has shaken hands with person b.

    (c) Given a set A, the relation $\sim$ on $\mathcal{P}(A)$ given by $A_1 \sim A_2$ if and only if $A_1 \subseteq A_2$.

11. Let $A = \{a, b, c, d\}$. For each of the following, find the smallest equivalence relation containing the given ordered pairs. By "find", we mean that you should give both the underlying partition **and** a complete listing of all ordered pairs in the relation. (each part is a separate presentation problem).

    (a) $A_1 = \{(a, a), (a, b), (b, c), (d, d)\}$

    (b) $A_2 = \{(a, a), (b, a), (c, d)\}$

    (c) $A_3 = \{(a, b), (a, c), (a, d)\}$

    (d) $A_4 = \{(a, b)\}$

12. This problem involves a famous set-theoretic concept called Russell's Paradox. According to Russell's definitions, a set A is called **normal** if A is *not* an element of itself. Similarly, a set is **abnormal** if it *is* an element of itself.

    (a) Give an example of a set that is normal.

    (b) Give an example of a set that is abnormal.

    (c) Let $\mathcal{N} = \{A \,|\, A \text{ is a set that is normal }\}$ and let $\mathcal{A} = \{A \,|\, A \text{ is a set that is abnormal }\}$.
    Is $\mathcal{N}$ a normal set or an abnormal set? How about $\mathcal{A}$? Explain how this leads to a paradox and comment on what caused things to go wrong.

Part 2:

The following problems have been compiled to both aid you in preparing for the final exam and to serve as a list of presentation eligible problems during our last regular class meeting on Tuesday, May 7th. You are expected to give a clear and correct proof, and to be able to talk your classmates through your argument as you are presenting it on the board.

1. Find the logical negation of the statement: Everyone who passed the exam studied for the exam.
2. Determine whether or not the given argument is valid by either giving a proof or by finding a counterexample.

   If I work hard every day then I will get a promotion at work.
   If I do not get a promotion at work then I will not be able to afford my house payment.
   I can afford my house payment.

   Therefore I work hard every day.

3. Determine whether or not the given argument is valid by either giving a proof or by finding a counterexample.

   If I want to go out on Saturday night then I need to study for my exam during the afternoon.
   I do not want to go out on Saturday night.

   Therefore I do not need to study for my exam during the afternoon.

4. Determine whether or not the given argument is valid by either giving a proof or by finding a counterexample.

   If gas is expensive and parking is inconvenient then I will take the bus to school.
   If I take the bus to school then I will not be able to take a night class.
   I am taking a night class.

   Therefore gas is not too expensive or parking is convenient.

5. Prove that if n is an integer and that $n^2 + 11$ is even, then n is odd.
6. **Prove or Disprove:** Let a, b, and c be integers. If ab divides c, then a divides c.
7. Prove that for integers $n > 1$ that $n!$ is even. [Recall that $n! = n \cdot (n-1) \cdot \cdots 3 \cdot 2 \cdot 1$]
8. **Prove or Disprove:** The product of any 3 consecutive integers is a multiple of 3.
9. Determine whether or not there exist integers m and n such that $2m + 4n = 7$.
10. Prove that if r^2 is irrational, then r is also irrational.
11. Prove that $A - B = A \cap \overline{B}$.
12. Use Proof by contradiction to prove the following: If A and B are sets, then $A \cap (B - A) = \emptyset$.
13. **Prove or Disprove:** If A or B are sets satisfying $A \cup B = A \cap B$, then $A \cap \overline{B} = \emptyset$.
14. Prove that $n^5 - n$ is divisible by 5 for any non-negative integer n.
15. Prove that for all n, $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$
16. Suppose that $f(x) = e^x$ and $g(x) = xe^x$. Use induction and the product rule to show that $g^{(n)}(x) = (x+n)e^x$ for all $n \geq 1$.
17. Let f be the function defined by $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ $f(m, n) = m^2 - n$. Determine whether or not f is a one-to-one. Also determine whether or not f is onto. Justify your answers.
18. Suppose $R = \{(x, y) \in \mathbb{R}^2 : y - x \text{ is an integer } \}$. Prove that R is an equivalence relation.
19. Define a relation R on $\mathbb{R}^2$ by $\{((x_1, y_1), (x_2, y_2)) \mid (x_1^2 + y_1^2) = (x_2^2 + y_2^2)\}$
    (a) Show that R is an equivalence relation.
    (b) Describe the equivalence classes of R.
20. Let A be the set of all people who attend MSUM. Define a relation on A as follows: $(a, b) \in \mathcal{R}$ if person a has completed at least as many credits as person b. Determine whether or not $\mathcal{R}$ is a poset.