> **"The only way we wouldn't make money was if the game was fixed—in which case we'd make even more money."**

Mixing a **Night out with Probability...** & Making a Fortune

Kari Lock
Williams College

When most people kick back for a beer in a bar, they probably don't see a lottery ticket and think "Hypergeometric Distribution!" However, two former probability students, curious as to how much the state was making, did just that. While in a bar one day, they took a Quick Draw lottery ticket (a popular bar game sponsored by the New York State Lottery) and decided to calculate the estimated payoff on the dollar. It wasn't until a year and a half later that their curiosity paid off… big time.

Let's take a hypothetical Friday night and pretend you are spending the evening out with your buddies at a local bar. On the edge of your table are Quick Draw tickets, displaying the numbers 1-80, with little boxes under each where you can check them off. You have the option of selecting anywhere between 1 and 10 of these numbers (or having your numbers randomly picked for you). There is a colorful TV screen across the room, and every five minutes, twenty new numbers randomly selected by the state appear on the screen. If you are feeling lucky this night out, you would pick your numbers, decide how much you want to bet, and then eagerly wait. As the new numbers appear on the screen, you would count the number of matches there are between those and your picks, and this number of matches would determine your payoff.

That is what you would do if you were a typical person out for a drink. However, maybe you just took a course in probability and remember problems from your homework on hypergeometric distributions, which result from sampling without replacement.

You start off with a set of size n, (the set of numbers 1–80), and out of this set you choose a subset of size m (the ten numbers you pick on your ticket). Out of the original set, there are r "successes" (the 20 numbers selected by the state), and you want to find the probability of some number x of these successes being contained in your subset (the probability of your picks having x matches with those on the screen). The payoff per dollar for each number of matches is given on your ticket, so once you have the probability of getting each possible number of matches, it is a simple matter to calculate what you really care about, the expected payoff.

The probability of getting x matches when choosing m numbers, $p(x)$, is found using binomial coefficients. We first need to compute the number of possible combinations for observing x matches, given a subset of size m. This is given by

$$\binom{r}{x}\binom{n-r}{m-x}.$$

The first term is the number of ways to choose the x successes for your subset from the r total successes, and for each of these possibilities, there are all the ways to choose the $m-x$ failures for your subset out of the $n-r$ total failures. Thus the product gives you the total number of ways to choose a subset of size m containing x matches. To find $p(x)$, we need to divide this quantity by the total number of possibilities for choosing a subset of size m (regardless of the number of matches). This is given by

$$\binom{n}{m}.$$

We thus observe

$$p(x)=\frac{\binom{r}{x}\binom{n-r}{m-x}}{\binom{n}{m}}.$$

From here, it is a simple matter of plugging in numbers… calculate $p(x)$ for each possible x, from which you can find the probability of each payoff, and then find the expected payoff for your dollar.

In a small town in New York State, when Quick Draw was introduced to the local bar, two former probability students did just this. They saw the ticket and made the connection between the lottery game and the hypergeometric distribution. Once the connection was made, one of them "just took out his old textbook, found the hypergeometric distribution, and plugged in the numbers." For example, choosing 4 numbers (referred to in bar lingo as "4 Spot"), they simply had to compute the probability of getting 0, 1, 2, 3, or 4 matches. The probability of getting 1 match, $p(x)$, is worked out below:

$$p(1) = \frac{\binom{20}{1}\binom{60}{3}}{\binom{80}{4}} = \frac{\frac{20}{1}\cdot\frac{60\cdot 59\cdot 58}{3\cdot 2\cdot 1}}{\frac{80\cdot 79\cdot 78\cdot 77}{4\cdot 3\cdot 2\cdot 1}} = .433.$$

The remaining probabilities can be computed without much difficulty, resulting in $p(0) = .308$, $p(1) = .433$, $p(2) = .213$, $p(3) = .043$, and $p(4) = .003$. The game ticket in Figure 1 shows the payoff per dollar bet for each number of matches.

Thus the expected payoff can be computed by summing over the probability of each number of matches times the payoff for that number of matches. So we find the expected payoff per dollar for the 4 Spot game to be .308(0) + .433(0) + .213(1) + .043(5) + .003(55) = \$0.59. Since this is the payoff per dollar bet, any intelligent person will realize that odds are you will lose money playing 4 Spot Quick Draw.

However, about one year later one of these guys was back at the bar and read an advertisement for Quick Draw—an ad stating that every Wednesday in November, payoff on the 4 Spot game is doubled! Having already utilized his probability background and computed the average payoff to be about 60¢ for the dollar (as we did above), this guy saw the ad doubling this payoff, and immediately thought "Holy ****!!! They're giving away money!" And it was immediately clear that these students were going to earn more than just an "A" from their probability class.

Together they worked out the number of times they would have to play in order to ensure making a profit. When asked about his confidence prior to playing, one of them commented "I was very confident. The sample size was large enough so that the only way we wouldn't make money was if the game was fixed—in which case we'd make even more money." They had no fear going into the event, for they both had total faith in statistics, probability, and the law of large numbers.

When the bar opened at 10 am the first Wednesday in November, they were there and ready to go. From opening until the deal expired at midnight, for all four Wednesdays in November, these two guys feverishly played 4 Spot Quick Draw. Purchasing around 1500 tickets a day, they played the maximum amount of 20 games with each ticket, betting \$5 a game. As they played more and more games, they started making a profit as predicted, and were able to use their winnings to keep purchasing more tickets. The only factors limiting the number of tickets they played were the printer—it took a certain amount of time for the machine to process and print out a ticket—and the actual process of cashing in the tickets. As for the colorful monitor displaying the results, it could have been turned off for all the guys cared. They were so confident in the outcome and so busy trying to maximize the number of tickets played that they didn't even pause to observe the results on the screen overhead. It turns out that their faith in probability was justified. Their final profit after the four days of playing ended up within \$100 of what they had originally computed to be their expected payoff.

By the fourth Wednesday in November people were starting to realize what was going on, and several groups monopolized the Quick Draw games in bars around the county. One man, deciding to capitalize on the success of others without bothering to work out the math himself, ran into trouble. He spent the whole day playing 5 Spot Quick Draw (which didn't have the double the payoff special). I'm sure you can predict the result for this unfortunate man.

For the original two students, however, their knowledge of probability and their initiative to apply it proved to be extremely fruitful. After purchasing a new house and a new car, one of the guys was asked to comment on the experience. His words of wisdom after the whole event: "It shows that paying attention in math class can, in fact, be useful."

So exactly how rich did the combination of probability and curiosity make these guys? Well… you do the math. ■

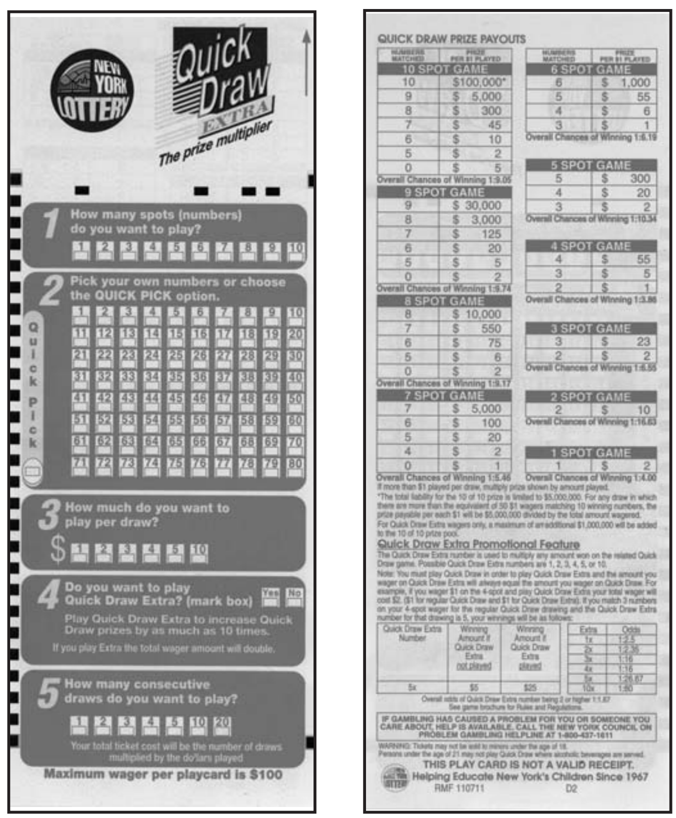

Figure 1. A New York State Lottery Quick Draw Ticket. Note the portion corresponding to the payoff amounts for the 4 Spot Game.