Dan Kalman and Maurice Burke

Solving Cubic Equations with Curly Roots

To the casual observer, there appears to be little difference between

$$x^2 + bx + c = 0 \quad (1)$$

and

$$x^3 + bx + c = 0. \quad (2)$$

But appearances can be deceiving. The first equation is immediately familiar to mathematics teachers, who know that its solutions are given by the quadratic formula as

$$\frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

A corresponding formula for cubics exists, but it is much more complicated. Rare is the teacher who can retrieve it from memory, obtaining a solution to (2) as

$$\sqrt[3]{\frac{-c}{2} + \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}} + \sqrt[3]{\frac{-c}{2} - \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}}.$$

Moreover, although this "Cardano formula" always yields at least one real solution, its calculation sometimes requires extracting the cube root of a complex number, an unavoidable complication.

Philosophically, this generalization from quadratic to cubic equations follows an obvious path. Because square roots provide solutions to quadratics, the impulse to use cube roots to solve cubic equations is altogether natural. But a less natural approach leads to a much more elegant result, giving a solution of (2) as

$$x = (-c/b)\ \{\overline{b^3/c^2}, \quad (3)$$

where the notation $\{\overline{t}$ represents the *curly root* of t. As we shall see, curly roots and cube roots are closely related, but curly roots are better suited to the purpose of solving cubic equations.

The idea behind curly roots appears to have been discovered independently by Nogrady (1937) and Pettit (1947), although the terminology and notation used here were introduced in Kalman (2009). In a time when technology enables students to create and manipulate functions and their graphs, the study of functions such as curly root becomes feasible in the modern curriculum.

THE DEFINITION OF CURLY ROOT

Let $F(x) = x^3/(1 - x)$ for $x < 1$. As indicated in **figure 1**, F is increasing on its domain and, hence,

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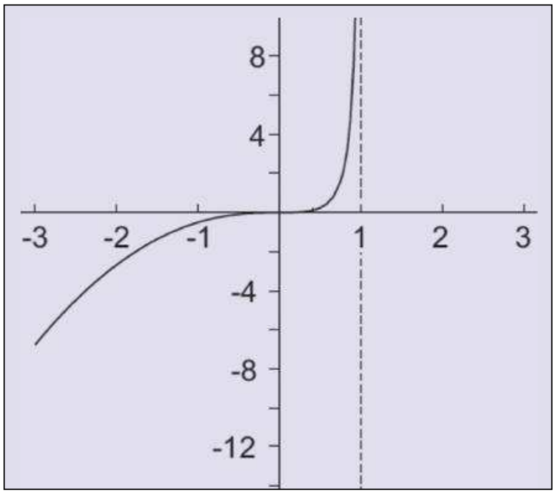

Fig. 1 The increasing function $F(x) = x^3/(1 - x)$ for $x < 1$ is graphed.

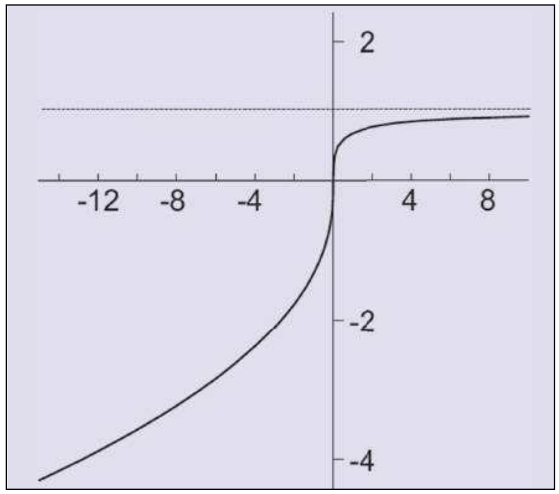

Fig. 2 The curly root function $y = \{\overline{x}$ is graphed as the inverse of $F(x)$.

is invertible. We define the curly root function to be the inverse of F. That is, for any real x, $\{\overline{x}$ is defined by

$$y = \{\overline{x} \text{ if and only if } x = y^3/(1 - y) \text{ where } y < 1.$$

Defining the curly root in this way immediately gives us its graph (shown in **fig. 2**) and the two identities

$$F(\{\overline{x}) = x \text{ for all real } x \qquad (4)$$

and

$$\{\overline{F(x)} = x \text{ for all } x < 1. \qquad (5)$$

Although $\{\overline{x}$ is not a familiar function, there is nothing strange about the way it is defined. Consider this analogous definition:

$$y = \sqrt{x} \text{ if and only if } x = y^2 \text{ where } y \geq 0.$$

The fact is that many familiar functions, including radicals and logarithms, are defined in terms of an inverse function. Further, the curly root function is easily defined, tabulated, graphed, and saved in a calculator's tool kit of functions (see **figs. 3–5**). From these perspectives, the curly root function is commonplace.

Before demonstrating how curly roots can be used to solve cubic equations, we make some simple observations about the nature of the curly root function.

PROPERTIES OF CURLY ROOT

We can obtain exact values of the curly root function using its definition as an inverse function, just as we do for the square root or logarithm functions. For example, since $F(0) = 0$, we know that $\{\overline{0} = 0$. Similarly, $F(-1) = -1/2$, so $\{\overline{-1/2} = -1$. One special value with particular significance is $\{\overline{-27/4} = -3$, as we shall see later.

Other properties of the curly root can also be inferred using its definition as an inverse. Consider the following properties of F, as illustrated in **figure 1**:

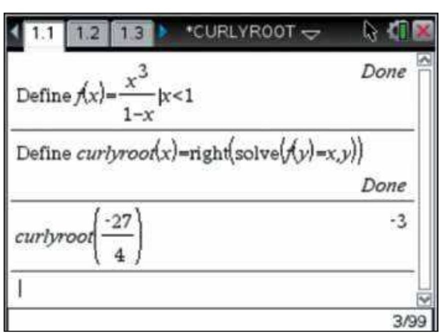

Fig. 3 A function equation with x- and y-values interchanged is solved for y to define the curly root function on the TI-Nspire CAS.

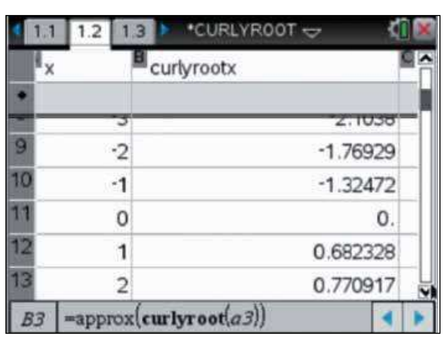

Fig. 4 A table of curly root values is easily created.

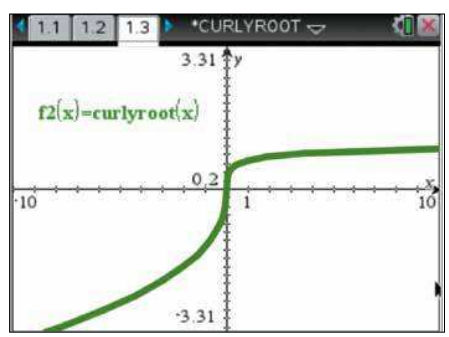

Fig. 5 The curly root function's graph is obtained directly from its definition.

- It is differentiable on its domain, with $F'(0) = 0$ and $F'(x) > 0$ for $x \neq 0$.
- It is increasing on its domain.
- $F(x) \to \infty$ as $x \to 1$, so the graph has a vertical asymptote at $x = 1$.
- $F(x) \to -\infty$ as $x \to -\infty$.
- The graph is concave down for $x < 0$, is concave up for $x > 0$, and has an inflection point at $x = 0$.

These observations are readily confirmed analytically using standard methods of calculus. They lead, in turn, to corresponding properties of curly root:

- It is differentiable for all x except $x = 0$, and the derivative is positive wherever it is defined.
- It is increasing on its domain.
- $\{\overline{x} \to 1$ as $x \to \infty$, so the graph has a horizontal asymptote at $y = 1$.
- $\{\overline{x} \to -\infty$ as $x \to -\infty$.
- The graph is concave up for $x < 0$, is concave down for $x > 0$, and has an inflection point at $x = 0$.

These characteristics contribute to our general knowledge about the curly root function. As one specific application, the fact that $\{\overline{x}$ is an increasing function tells us that it preserves inequalities. That is, if $u < v$, then $\{\overline{u} < \{\overline{v}$.

SOLVING THE REDUCED CUBIC

The general form of a cubic equation is $x^3 + ax^2 + px + q = 0$, where a, p, and q are real constants. The substitution $x = z - a/3$ always transforms this equation to one with no quadratic term: $z^3 + bz + c = 0$, a so-called *reduced cubic equation*. Consequently, to find roots of all cubics, it is sufficient to find roots of the reduced cubic. For the remainder of this article, therefore, we consider only the reduced cubic equation (2). Our immediate goal is to verify that (3) yields a solution to (2). This is easy in principle: Just substitute the proposed value of x into the original equation and show that it yields a solution. In practice, first deriving another identity makes life a little simpler. From equation 4 and the definition of F, we have

$$\frac{\left(\{\overline{x}\right)^3}{1 - \{\overline{x}} = x$$

for all x. Rearrangement produces

$$(\{\overline{x})^3 = x - x\{\overline{x}. \qquad (6)$$

Now let's verify that $x = (-c/b)\{\overline{b^3/c^2}$ gives a solution to $x^3 + bx + c = 0$. We must show that

$$[(-c/b)\{\overline{b^3/c^2}]^3 + b(-c/b)\{\overline{b^3/c^2} + c = 0.$$

In the first term, which can be rewritten as $(-c^3/b^3)[\{\overline{b^3/c^2}]^3$, we apply (6), obtaining

$$(-c^3/b^3)[b^3/c^2 - b^3/c^2\{\overline{b^3/c^2}] + b(-c/b)\{\overline{b^3/c^2} + c.$$

Simplifying produces

$$-c + c\{\overline{b^3/c^2} + -c\{\overline{b^3/c^2} + c.$$

This is equal to 0, which is what we wished to show.

Although this process confirms the validity of (3), it sheds little light on how that formula was discovered or how the idea of curly root might have originated. A more revealing approach presents itself if we make a change of variables in (2). For example, if we replace x with $2u$, the equation becomes

$$8u^3 + 2bu + c = 0,$$

and dividing by 8 yields

$$u^3 + \frac{b}{4}u + \frac{c}{8} = 0.$$

This process shows that we can transform one reduced cubic equation to another with modified coefficients. In fact, as the example suggests, replacing x with su has the effect of dividing b by s^2 and dividing c by s^3. Of all the different versions of the equation that we might form, is one simpler than the rest?

One possibility is to make the coefficients b and c equal. That is, choose s so that $b/s^2 = c/s^3$. This can be accomplished by taking $s = c/b$, so our substitution is $x = cu/b$. Then (2) is transformed into

$$u^3 + \frac{b^3}{c^2}u + \frac{b^3}{c^2} = 0$$

or, introducing $a = b^3/c^2$, into $u^3 + au + a = 0$. Here we catch a glimpse of the function F. Rewriting the equation so that a is isolated, we obtain

$$a = \frac{-u^3}{1+u} = \frac{(-u)^3}{1-(-u)} = F(-u).$$

In terms of curly root, this means that $-u = \{\overline{a}$ and, hence, $u = -\{\overline{a}$. Now replacing a with b^3/c^2 and u with bx/c, we can rederive the curly root solution (3). Perhaps it was just such an analysis that first revealed the significance of F for solving cubic equations.

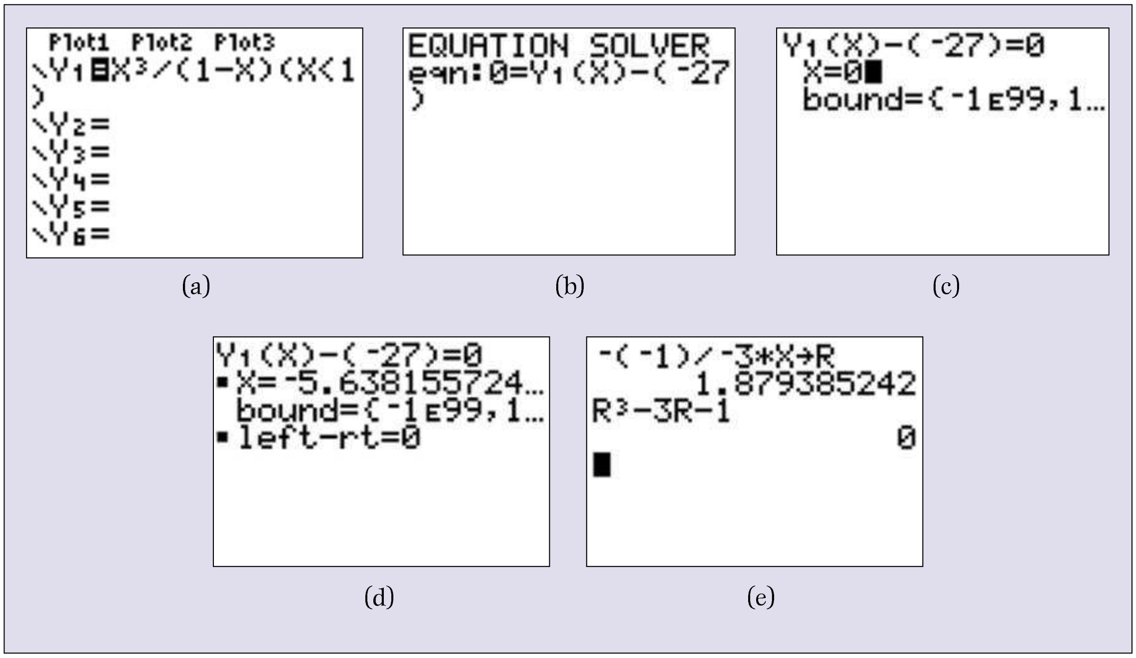

Fig. 6 Computing curly roots is easy on the TI-84.

HOW MANY REAL ROOTS?

Results such as the quadratic formula or the curly root solution of the cubic have significance beyond computing solutions to specific equations. They can be manipulated algebraically to derive additional results. As one example, let's use (3) to determine when a reduced cubic has only real roots.

Consider a reduced cubic $p(x) = x^3 + bx + c$ with $b, c \neq 0$. Let r be the root given by (3). That is, $r = (-c/b)\{\overline{b^3/c^2}$. Since $(x - r)$ must be a factor of $p(x)$, by dividing we obtain $p(x) = (x - r)(x^2 + rx + r^2 + b)$. Thus, p has all real roots if and only if the quadratic factor does. That, in turn, is determined by the sign of the discriminant $D = r^2 - 4(r^2 + b) = -3r^2 - 4b$. In particular, we would like to derive necessary and sufficient conditions for D to be nonnegative.

First, because r is not 0, $D = (-3r^3 - 4br)/r$. Next, since $p(r) = 0$, $r^3 = -br - c$, and so $D = (3c - br)/r = 3c/r - b$. Substituting $r = (-c/b)\{\overline{b^3/c^2}$, we have $D = -b(1 + 3/\{\overline{b^3/c^2})$. If $b > 0$, then $b^3/c^2 > 0$, so $\{\overline{b^3/c^2} > 0$. In this case, $D = -b(1 + 3/\{\overline{b^3/c^2}) < 0$, and the cubic has a unique real root. If $b < 0$, then $D \geq 0$ if and only if $(1 + 3/\{\overline{b^3/c^2}) \geq 0$ or, equivalently, $\{\overline{b^3/c^2} \leq -3$.

Thus, the reduced cubic equation (2) has three real roots if and only if $\{\overline{b^3/c^2} \leq -3$.

Further, because the curly root function is increasing, $\{\overline{b^3/c^2} \leq -3$ holds if and only if $b^3/c^2 \leq -27/4$. So we have shown that the reduced cubic has all real roots exactly when $b^3/c^2 + 27/4 \leq 0$. This inequality is algebraically equivalent to $b^3/27 + c^2/4 \leq 0$, recognizable as the traditional discriminant condition for the reduced cubic, with equality indicating repeated roots.

AN EXAMPLE

We illustrate the use of curly roots for solving cubic equations first with a TI-84 calculator and then with a TI-Nspire CAS. Consider the equation $x^3 - 3x - 1 = 0$. With $b = -3$ and $c = -1$, one solution to the equation is $(-c/b)\{\overline{b^3/c^2}$, so we need to compute the curly root of –27. With a TI-84 calculator, we define the inverse of the curly root function in Y1 (see **fig. 6a**). In the **MATH** menu **Solver** option, we enter $0 = Y1(x) - (-27)$, since the curly root of –27 is the x-value that satisfies this equation (see **fig. 6b**). The TI-84 Solver uses a modified Newton's method to find solutions of equations and so depends on initializing the x-value to something reasonably close to the solution. Setting $x = 0$ works well (see **fig. 6c**). Selecting **ALPHA** and then **ENTER**, we get –5.638155724 as the curly root of –27 (see **fig. 6d**). Because this is less than –3, the curly root discriminant criterion tells us that there are three real solutions. Finally, we calculate $-(-1)/(-3)x$ in the main window to get the curly root solution, $R = 1.879385242$, and verify that R is a solution (see **fig. 6e**).

We hasten to point out that Solver, applied directly to $x^3 - 3x - 1 = 0$, yields all three solutions, using $x = -2$, $x = 0$, and $x = 2$ as initial values for X, with the solution shown in **figure 7**. This raises the question: Why bother with curly roots? One answer is that the curly root formula leads to general results about cubic equations, such as the curly root discriminant criterion that we just applied. Additional answers will be provided later.

Finding the curly root solution to $x^3 - 3x - 1 = 0$ with the TI-Nspire CAS is more straightforward.

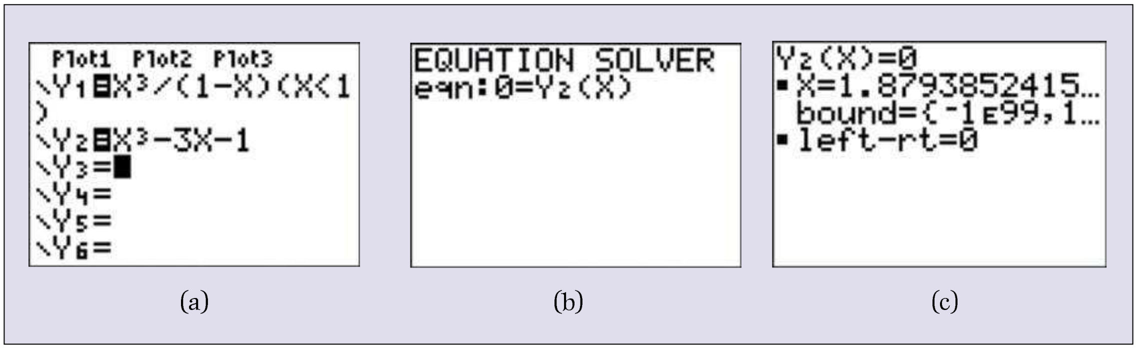

Fig. 7 **SOLVER** yields approximate solutions for equations.

Once the curly root function is defined (see **fig. 3**), we simply enter the curly root formula (see **fig. 8a**). The Nspire supplies an exact symbolic solution whose approximate value agrees with what was given by the TI-84. But the symbolic solution compels the question: Where do those trigonometric functions come from?

Using basic trigonometric identities, we can reduce the solution $\cos(2\pi/9) + \sin(2\pi/9)(\sqrt{3})$ to $2\cos(\pi/9)$ or $2\cos(20°)$. It is well known in the history of mathematics, thanks to the French mathematician Vieta, that when $x^3 + bx + c = 0$ has three real solutions, then $b < 0$ and the three solutions have the form $p\cos(r)$, $p\cos(r + 120°)$, and $p\cos(r + 240°)$, where $p = \sqrt{-4b/3}$ and $\cos(3r) = (-c/2)\sqrt{-27/b^3}$ (Martin 1998). We verify the latter two solutions for our particular equation (where $p = 2$ and $r = 20°$) in **figure 8b**, recognizing –2E⁻13 as essentially zero.

PUTTING CURLY ROOTS INTO PERSPECTIVE: A HISTORICAL EXAMPLE

We intentionally chose $x^3 - 3x - 1 = 0$ in the example above. This equation is the result of substituting $\theta = 20°$ and then $\cos(20°) = x/2$ into the trigonometric identity $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$. The identity is used in the most common algebraic proof that Euclidean straightedge and compass constructions are not sufficient for trisecting a 60° angle. With modern methods, solving a cubic can be reduced to trisecting an angle, a problem from classical Greece that has inspired interest right down to the present day (Dudley 1996).

Our curly root development echoes a historical solution of the trisection problem. The earliest recorded mathematically defined curve that was not part of a line or a circle was the quadratrix, a curve apparently developed by Hippias (ca. 420 BCE) specifically for dividing an angle into any number of equal parts (Beckmann 1971, pp. 40–44). Like the curly root function, the quadratrix curve can be represented as an inverse function, $y = Q^{-1}(x)$, where $Q(x) = x\cot(x\pi/2)$ for $0 \le x \le 1$ (see **fig. 9**). This is the modern equivalent of Pappus's definition of the quadratrix.

The quadratrix did not arise accidentally but, like curly root, was designed with a specific use in mind. In **figure 9b**, segment OC intersects the quadratrix curve at D, and E is the foot of the perpendicular from E on segment OI. The crucial property of the quadratrix that allows it to be used to easily partition any angle into any number of equal parts is EO/IO = length(arc BQC)/length(arc BQI). Thus, by trisecting the segment OE we can construct the trisector of $\angle COB$.

This historical example makes the point that, from earliest times, the "solving process" in mathematics has frequently involved defining functions to help address specific problems. The narrative of

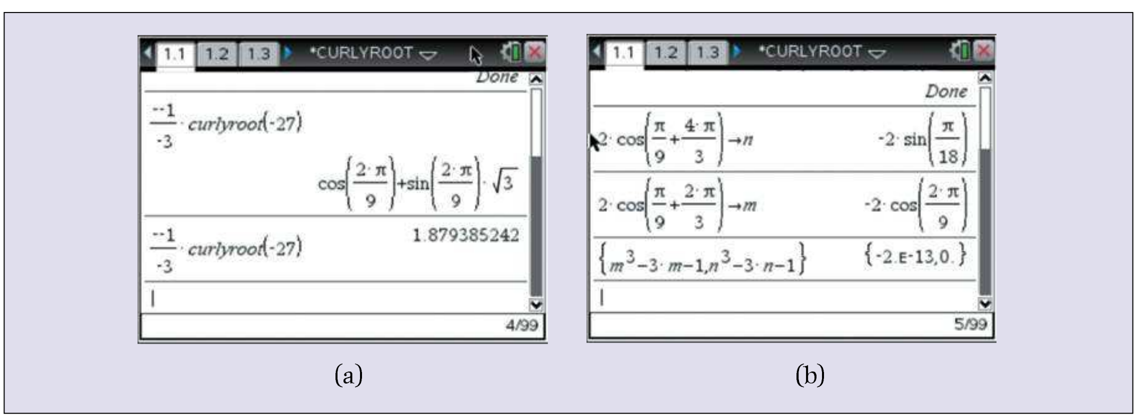

Fig. 8 Curly roots are verified using a TI-Nspire CAS.

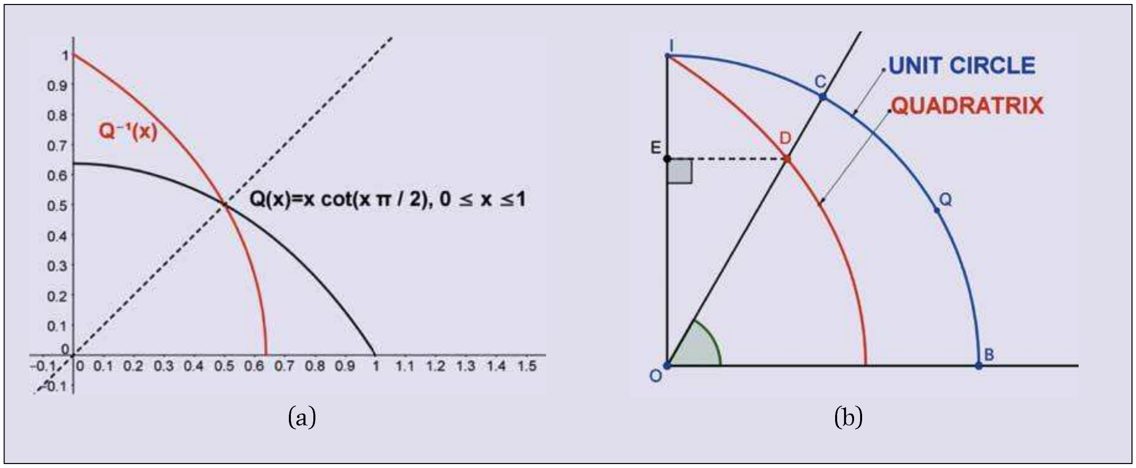

Fig. 9 The quadratrix curve is used to trisect arbitrary angles.

mathematics is replete with such examples, some named but most unnamed. Those that have found a place in the core mathematics curriculum are so familiar that we take them for granted. Yet these functions were developed and found application in much the same way as their more exotic cousins, such as the quadratrix, ultraradicals, and curly roots. Indeed, our development of the curly root closely resembles standard treatments of square root and logarithmic functions. Each case involves the inverse of one specific function (e.g., $y = x^2$, $y = e^x$, or $y = x^3/(1 - x)$). And just as square roots permit the solution of any quadratic equation and logarithms the solution of any exponential equation, so too curly roots permit us to solve any cubic equation. Following an analogous development, can you invent a function for solving equations such as $2^x = 3x + 4$? (See Kalman 2001).

THE SOLVING PROCESS

Great value is placed on numerical answers to computational questions in the mathematics curriculum. However, if the goal is only to have our students think of the "solving" process as finding numerical answers, then we could certainly use the TI-84 Solver to replace most procedures currently taught for solving equations. But our goals are much broader. In addition to understanding some basics of solving equations, we want students to develop a sense of the solving process, the role that functions play in this process, and why some functions that they encounter in the curriculum are deemed important. In short, we want students to embrace creating their own functions, thinking with functions, and even giving them names—like the curly root.

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DAN KALMAN, kalman@american.edu, formerly worked as an applied mathematician in the aerospace industry and teaches mathematics at American University in Washington, DC. **MAURICE BURKE,** Maurice.Burke@utsa.edu, has recently retired from Montana State University and The University of Texas at San Antonio. He continues to remain interested and involved in mathematics education and, of course, *Mathematics Teacher*.