Helping Undergraduates Learn to Read Mathematics

Ashley Reiter, Maine School of Science and Mathematics, 77 High Street Limestone, ME 04750, reitera@mssm.lcs.k12.me.us
(https://www.maa.org/node/121566 -- accessed on 2/10/2019)

Although most students "learn to read" during their first year of primary school, or even before, reading is a skill which continues to develop through primary, secondary and post-secondary school, as the reading material becomes more sophisticated and as the expectations for level of understanding increase. However, most of the time spent deliberately helping students learn to read focuses on literary and historical texts. Mathematical reading (and for that matter, mathematical writing) is rarely expected, much less considered to be an important skill, or one which can be increased by practice and training.

Even as an undergraduate mathematics major, I viewed mathematical reading as a supplementary way of learning--inferior to learning by lecture or discussion, but necessary as a way of "filling in the gaps." Not until graduate school was I responsible for reading new material at a high level of comprehension. And, as I began to study primarily written mathematics (texts and articles) rather than spoken mathematics (lectures), I discovered that the activities and habits needed to learn from written mathematics are quite different from those involved in learning from a mathematics lecture or from those used in reading other types of text. As I consciously considered how to read mathematics more effectively and to develop good reading habits, I observed in my undergraduate students an uneasiness and lack of proficiency in reading mathematics.

In response to this situation, I wrote for my students (mostly math majors in Introductory Abstract Algebra at the University of Chicago) two handouts, one on reading theorems and the other on reading definitions. These describe some of the mental activities which help me to read mathematics more effectively. I also gave a more specific written assignment, applying some of these questions to a particular section of assigned reading. My hope was that, as they were forced to actively engage in reading, they would discover that reading mathematics could be a profitable pursuit, and that that they would develop habits which they would continue to use.

More than one such written exercise is needed to significantly affect the way that students view reading. While the students seemed to understand the types of questions that are helpful, they needed some practice in carrying these out, and even more practice using these activities in the absence of a written assignment.

A Few Mathematical Study Skills... Reading Theorems

In almost any advanced mathematics text, theorems, their proofs, and motivation for them make up a significant portion of the text. The question then arises, how does one read and understand a theorem properly? What is important to know and remember about a theorem?

A few questions to consider are:

- What kind of theorem is this? Some possibilties are:
    - A classification of some type of object (e.g., the classification of finitely generated abelian groups)
    - An equivalence of definitions (e.g., a subgroup is normal if, equivalently, it is the kernel of a group homomorphism or its left and right cosets coincide)
    - An implication between definitions (e.g., any PID is a UFD)
    - A proof of when a technique is justified (e.g., the Euclidean algorithm may be used when we are in a Euclidean domain)
    - Can you think of others?

- What's the content of this theorem? E.g., are there some cases in which it is trivial, or in which we've already proven it?
- Why are each of the hypotheses needed? Can you find a counterexample to the theorem in the absence of each of the hypotheses? Are any of the hypotheses unneccesary? Is there a simpler proof if we add extra hypotheses?
- How does this theorem relate to other theorems? Does it strengthen a theorem we've already proven? Is it an important step in the proof of some other theorem? Is it surprising?
- What's the motivation for this theorem? What question does it answer?

We might ask more questions about the proof of theorem. Note that, in some ways, the easiest way to read a proof is to check that each step follows from the previous ones. This is a bit like following a game of chess by checking to see that each move was legal, or like running the spell checker on an essay. It's important, and necessary, but it's not really the point. It's tempting to read only in this step-by-step manner, and never put together what actually happened. The problem with this is that you are unlikely to remember anything about how to prove the theorem. Once you've read a theorem and its proof, you can go back and ask some questions to help synthesize your understanding. For example:

- Can you write a brief outline (maybe 1/10 as long as the theorem) giving the logic of the argument -- proof by contradiction, induction on n, etc.? (This is KEY.)
- What mathematical raw materials are used in the proof? (Do we need a lemma? Do we need a new definition? A powerful theorem? and do you recall how to prove it? Is the full generality of that theorem needed, or just a weak version?)
- What does the proof tell you about *why* the theorem holds?
- Where is each of the hypotheses used in the proof?
- Can you think of other questions to ask yourself?

A Few Mathematical Study Skills... Reading Definitions

Nearly everyone knows (or think they know) how to read a novel, but reading a mathematics book is quite a different thing. To begin with, there are all these definitions! And it's not always clear why one would care to know about these things being defined. So what should you do when you read a definition?

Ask yourself (or the book) a few questions:

- What kind of creature does the definition apply to? integers? matrices? sets? functions? some pair of these together?
- How do we check to see if it's satisfied? (How would we prove that something satisfied it?)
- Are there necessary or sufficient conditions for it? That is, is there some set of objects which I already understand which is a subset or a superset of this set?
- Does *anything* satisfy this definition? Is there a *whole class* of things which I know satisfy this definition?
- Does anything *not* satisfy this definition? For example?
- What special properties do these objects have, that would motivate us to make this definition?
- Is there a nice classification of these things?

Let's apply this to an example, abelian groups:

- What kind of creature does it apply to? Well, to groups... in particular, to a set together with a binary operation.
- How do we check to see if it's satisfied? The startling thing is that we have to compare every single pair of elements! This would be a big job, so:
- Are there necessary or sufficient conditions for it? Well, it's sufficient that the group be cyclic, as we saw in the homework. Do you know of any neccesary conditions?
- Does anything satisfy this definition? Well, yes... the group of rational numbers under addition, for example. We have a whole class of things which satisfy the definition, too -- cyclic groups.
- Does anything not satisfy this definition? Yes, matrix groups come to mind first. There *are* finite non-abelian groups, but this is harder to see... do you know of one yet?
- What special properties do these objects have, that would motivate us to make this definition? Some of these properties are obvious, others are things which we had to prove. One example: If H and K are subgroups of an abelian group, then HK is also a subgroup.
- Is there a nice classification of these things? Why, yes, at least for a large subcategory of them. We'll get to it later... it says, basically, that a finite abelian group is always built in a simple way from cyclic groups (Z_n's).