Math 127
06/18/2008

# Exponential Functions

**A. Definition:** An **exponential function** is a function of the form $f(x) = a^x$ for $0 < a < 1$ or $a > 1$.
Note: $a$ is called the **base** of the exponential function. As we will see, exponential functions have domain $(-\infty, \infty)$ and range $(0, \infty)$.

**Note:** The reason we exclude 0 and 1 as bases for exponential function is because $0^x = 0$ for and $x$, and $1^x = 1$ for any $x$, so these are just constant functions.

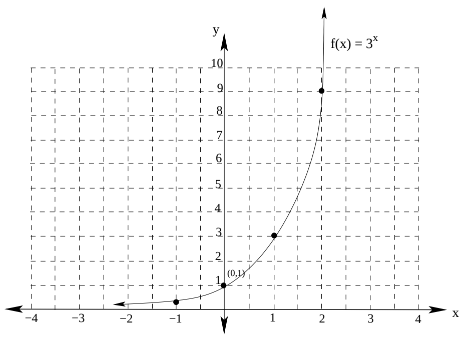
Notice that these functions are quite different from other functions we have looked at so far. Here, the *exponent* part of the expression defining the function is a *variable.*

**Example 1:** Let $f(x) = 3^x$. Then:
(a) $f(0) = 3^0 = 1$
(b) $f(2) = 3^2 = 9$
(c) $f(-3) = 3^{-3} = \frac{1}{27}$
(d) $f(\frac{2}{3}) = 3^{\frac{2}{3}} = \sqrt[3]{3^2} = \sqrt[3]{9} \approx 2.080084$

**Graphs of exponential functions:**

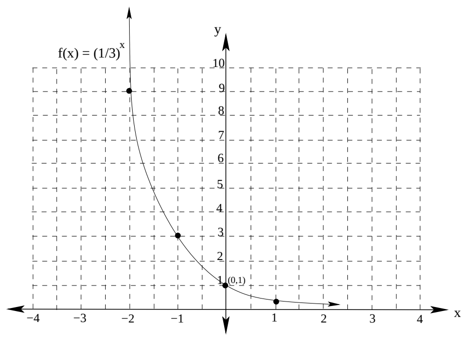
$f(x) = 3^x$

| $x$ | $f(x)$ |
|---|---|
| $-3$ | $\frac{1}{27}$ |
| $-2$ | $\frac{1}{9}$ |
| $-1$ | $\frac{1}{3}$ |
| $0$ | $1$ |
| $1$ | $3$ |
| $2$ | $9$ |
| $3$ | $27$ |




$f(x) = \left(\frac{1}{3}\right)^x$

| $x$ | $f(x)$ |
|---|---|
| $3$ | $\frac{1}{27}$ |
| $2$ | $\frac{1}{9}$ |
| $1$ | $\frac{1}{3}$ |
| $0$ | $1$ |
| $-1$ | $3$ |
| $-2$ | $9$ |
| $-3$ | $27$ |




**Facts:**
(1) If $f(x) = a^x$ with $a > 1$, then $f$ is an increasing function, and hence is a one-to-one function.
(2) If $f(x) = a^x$ with $0 < a < 1$, then $f$ is a decreasing function, and hence is a one-to-one function.

**Solving Basic Exponential Equations:**

We can use the fact that exponential functions are one-to-one to solve various equations involving exponentials. This is because we can make use of the fact that if $a^{x_1} = a^{x_2}$, then $x_1 = x_2$.
**Examples:**
1. $4^{2x-3} = 4^{5-x}$
Since $f(x) = 4^x$ is a one-to-one function, we can conclude that:
$2x - 3 = 5 - x$, or $3x = 8$.
Hence $x = \frac{8}{3}$.

2. $2^{4x-7} = 8^{2x-5}$
Since $8 = 2^3$, we can rewrite $8^{2x-5}$ as $\left(2^3\right)^{2x-5} = 2^{3(2x-5)} = 2^{6x-15}$.
Then, as above, we know that $4x - 7 = 6x - 15$, or $8 = 2x$.
Hence $4 = x$.

**Compound Interest**

There are many practical applications for exponential functions. One of the most common is computing compound interest.

**The Compound Interest Formula:** When a principal amount $P$ in invested at interest rate $r$ which is compounded $n$ times per year and remains invested for $t$ year, the amount $A$ that results is given by the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$

**Examples:**

1. Suppose you put \$1000 in an account that pays 6% interest compounded monthly. How much money will be in the account 3 years later?

   $P = 1000$, $r = 0.06$, $n = 12$, and $t = 3$, so $A = 1000\left(1 + \frac{.06}{12}\right)^{(12)(3)} = 1000\,(1.005)^{36} \approx \$1,196.68$

2. Now Suppose you put \$2000 in an account that pays 7% interest compounded daily. How much money will be in the account 5 years later?

   $P = 2000$, $r = 0.07$, $n = 365$, and $t = 5$, so $A = 2000\left(1 + \frac{.07}{365}\right)^{(365)(5)} \approx 2000\,(1.000191781)^{1825} \approx \$2838.04$

**The Natural Exponential Function:**

**Definition:** If we consider what happens to the base of our compound interest exponential term: $\left(1 + \frac{1}{n}\right)$ as we compound more and more frequently

| $n$ | $\left(1 + \frac{1}{n}\right)^n$ |
|---|---|
| 1 | 2.0 |
| 10 | 2.59374246 |
| 100 | 2.704813829 |
| 1,000 | 2.71692393 |
| 10,000 | 2.71814593 |
| 100,000 | 2.71826824 |
| 1,000,000 | 2.71828047 |

As $n$ gets bigger and bigger, $\left(1 + \frac{1}{n}\right)$ approaches an irrational number we call $e$, **the base of the natural exponential function**, $f(x) = e^x$. Since $e \approx 2.7182818$, $2^x < e^x < 3^x$.

**Continuously Compounded Interest** Using this new base, we can measure the accumulation of interest that is compounded "instantaneously" rather than only $n$ times a year. We do so using the formula: $A = Pe^{rt}$, where $P, A, r$, and $t$ are exactly as above.

**Example:** Suppose you invest \$1000 at 6% interest compounded continuously for 3 years. Then at the end of the 3 years, you will have: $1000e^{0.06(3)} \approx \$1,197.22$
Notice that this is about 54 cents more that we had investing the same amount at the same interest rate but only compounded monthly.

**Example:** Suppose the population of a bacterial colony if given by the function $f(t) = 500e^{-.87t}$ where $t$ is in hours and $f(t)$ is in thousands of cells.
Then $f(0) = 500e^{-.087(0)} = 500e^0 = 500$, so there are initially 500,000 cells in the colony.
Similarly, $f(5) = 500e^{-.087(5)} = 500e^{-0.435} \approx 323.632$, so after 5 hours, the population of the colony has been reduced to 323,632 cells.