The Inverse of a Matrix

A. Definitions:

1. The *identity matrix if size* n, denoted I_n, is a square matrix with 1s along the main diagonal and 0s everywhere else.

For example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. The *inverse* of a square $n \times n$ matrix A (if one exists) is an $n \times n$ matrix A^{-1} satisfying $AA^{-1} = A^{-1}A = I_n$

Note: The reason I_n is called the identity matrix (of size n) is because given any $n \times n$ matrix A, $A \cdot I_n = I_n \cdot A = A$. That is, I_n "acts like the number 1 " when multiplied with any $n \times n$ matrix.

Examples:

1. Let $A = \begin{bmatrix} 3 & 4 & -1 \\ 2 & 0 & 6 \\ 7 & -4 & 10 \end{bmatrix}$. Then $AI_n = \begin{bmatrix} 3 & 4 & -1 \\ 2 & 0 & 6 \\ 7 & -4 & 10 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & -1 \\ 2 & 0 & 6 \\ 7 & -4 & 10 \end{bmatrix}$

Similarly, $I_nA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & -1 \\ 2 & 0 & 6 \\ 7 & -4 & 10 \end{bmatrix} = \begin{bmatrix} 3 & 4 & -1 \\ 2 & 0 & 6 \\ 7 & -4 & 10 \end{bmatrix}$

2. Let $A = \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$. Then: $AB = \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 16-15 & 40-40 \\ -6+6 & -15+16 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and $BA = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 16-15 & -10+10 \\ 24-24 & -15+16 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus $B = A^{-1}$.

B. Finding the Inverse of a Matrix:

Given a square matrix A, we find its inverse (provided one exists) as follows:

1. Form the augmented matrix $[A|I]$
2. Use row operations to put the "A side" of this augmented matrix into reduced row echelon form.
3. If the resulting matrix has the form $[I|B]$, then $B = A^{-1}$

Examples:

1. Let $A = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$. Then the augmented matrix is: $\left[\begin{array}{cc|cc} 5 & 7 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right]$. We now reduce the augmented matrix using row operations:

$$\left[\begin{array}{cc|cc} 5 & 7 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_1-2R_2} \left[\begin{array}{cc|cc} 1 & 1 & 1 & -2 \\ 2 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{cc|cc} 1 & 1 & 1 & -2 \\ 0 & 1 & -2 & 5 \end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{cc|cc} 1 & 0 & 3 & -7 \\ 0 & 1 & -2 & 5 \end{array}\right]$$

Therefore, $A^{-1} = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$

2. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ We reduce the augmented matrix using row operations:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-3R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 3 & -2 & 0 \\ 0 & -1 & 2 & 2 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_3-R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 3 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 3 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_2-R_3]{R_1-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & -1 & 0 & 4 & -2 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]. \text{ Therefore, } A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

C. Solving Systems of Equations Using Inverse Matrices:
Now that we know how to find the inverse of a square matrix, if we are given a system of n equations in n unknowns, there is an alternative way to solve the system. We first need to translate our system of equations into a matrix equation of the form: $AX = B$. We do so as indicated by the following examples:

Example 1: Given the system of equations: $\begin{cases} 5x + 7y = 3 \\ 2x + 3y = -2 \end{cases}$

Let $A = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Then the matrix equation form for this system is:

$$\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Example 2: Given the system of equations: $\begin{cases} 2x + y + z = 4 \\ 3x + 2y + z = -1 \\ 2x + y + 2z = 0 \end{cases}$

Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $B = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$. Then the matrix equation form for this system is:

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$$

To solve a matrix equation of the form: $AX = B$, notice that if we multiply both sides of this equation on the left by A^{-1}, we get:
$A^{-1}AX = A^{-1}B$, or, by this inverse property of matrices, $IX = A^{-1}B$, or $X = A^{-1}B$. This new form allows us to "read off" the solution to the original system of equations.

Example 1: Again looking at the system of equations: $\begin{cases} 5x + 7y = 3 \\ 2x + 3y = -2 \end{cases}$, the matrix equation form for this system is:

$$\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Previously, we computed $A^{-1} == \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$.

Therefore, $X = A^{-1}B$ or $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 9 + 14 \\ -6 - 10 \end{bmatrix} = \begin{bmatrix} 23 \\ -16 \end{bmatrix}$

Check: $5(23) + 7(-16) = 115 - 112 = 3$, and $2(23) + 3(-16) = 46 - 48 = -2$

Example 2: Again looking at the system of equations: $\begin{cases} 2x + y + z = 4 \\ 3x + 2y + z = -1 \\ 2x + y + 2z = 0 \end{cases}$, the matrix equation form for this system is:

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$$

Previously, we computed $A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

Therefore, $X = A^{-1}B$ or $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 + 1 + 0 \\ -16 - 2 + 0 \\ -4 + 0 + 0 \end{bmatrix} = \begin{bmatrix} 13 \\ -18 \\ -4 \end{bmatrix}$

Check: $2(13) + (-18) + (-4) = 26 - 18 - 4 = 4$, $3(13) + 2(-18) + (-4) = 39 - 36 - 4 = -1$, and $2(13) + (-18) + 2(-4) = 26 - 18 - 8 = 0$