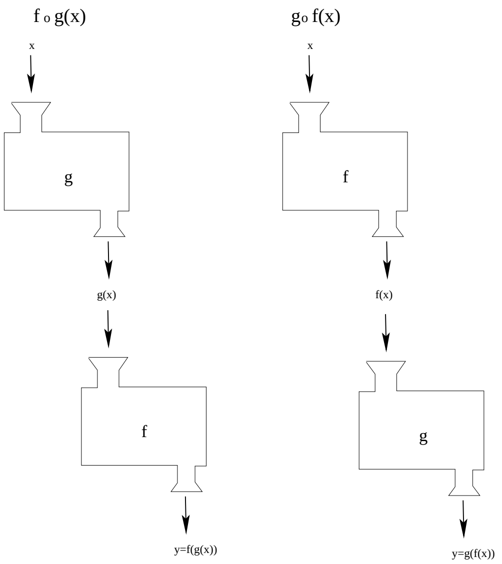
Math 229
The Algebra of Functions
06/03/2008

A. Building New Functions by Combining Old Ones

In practice, when we use functions to model real-world situations, we often create our model function by combining simpler functions. For example, when we built a function to model the monthly profit for our CD store, we first found functions modeling cost $C(x)$ and revenue $R(x)$. Then, we combined these to obtain a profit function $P(x) = R(x) - C(x)$. We used the subtraction operation in order to build a new function. We can build new functions from old ones using addition, subtraction, multiplication, and division.

Operation	**Formula**	**Domain**
Addition: $f + g$	$f + g(x) = f(x) + g(x)$	(Dom f)$\cap$(Dom g)
Subtraction: $f - g$	$f - g(x) = f(x) - g(x)$	(Dom f)$\cap$ (Dom g)
Multiplication: fg	$fg(x) = f(x) \cdot g(x)$	(Dom f)$\cap$(Dom g)
Division: $\frac{f}{g}$	$\frac{f}{g}(x) = \frac{f(x)}{g(x)}$	(Dom f)$\cap$(Dom g) $\cap \{x \mid g(x) \neq 0\}$

Another way of building new function from ones is a bit more complicated. The way we make the functions work together is to first put an input into one function. Then, we take the resulting output from the first function and we input it into the second function:



This process is called taking the **composition** of two functions, and the result of this process is called a **composite function**. Symbolically, we write composition as follows:

Operation	**Formula**	**Domain**
Composition: $f \circ g$	$f \circ g(x) = f(g(x))$	(Dom g) $\cap \{x \mid g(x) \in$ Dom $f\}$
$g \circ f$	$g \circ f(x) = g(f(x))$	(Dom f) $\cap \{x \mid f(x) \in$ Dom $g\}$

Examples:

1. Let $f(x) = 2x + 1$ and $g(x) = x^2 - 1$

   (a) $f + g(x) = (2x + 1) + (x^2 - 1) = x^2 + 2x$

   (b) $f - g(x) = (2x + 1) - (x^2 - 1) = -x^2 + 2x + 2$

   (c) $fg(x) = (2x + 1)(x^2 - 1) = 2x^3 + x^2 - 2x - 1$

   (d) $\frac{f}{g}(x) = \frac{2x+1}{x^2-1}$ [Domain: $x \neq \pm 1$]

   (e) $f \circ g(x) = f(g(x)) = 2(g(x)) + 1 = 2(x^2 - 1) + 1 = 2x^2 - 1$

   (f) $g \circ f(x) = g(f(x)) = (f(x))^2 - 1 = (2x + 1)^2 - 1 = 4x^2 + 4x$

2. Given that $h(x) = \sqrt{3x^2 + 4}$, write h, find functions f and g so that $h = f \circ g$.

   $f(x) = \sqrt{x}$ and $g(x) = 3x^2 + 4$
   or
   $f(x) = \sqrt{3x + 4}$ and $g(x) = x^2$
   or ... (there are many ways to do this)

3. Here an more complicated way of combining a function with itself. We will use this more later in the course:

   Let $f(x) = x^2 + 2x + 3$. Find $\dfrac{f(a+h) - f(a)}{h}$.

   $f(a) = a^2 + 2a + 3$

   $f(a + h) = (a + h)^2 + 2(a + h) + 3 = x^2 + 2ah + h^2 + 2a + 2h + 3$, so

   $$\frac{f(a+h) - f(a)}{h} = \frac{(a^2 + 2ah + h^2 + 2a + 2h + 3) - (a^2 + 2a + 3)}{h}$$
   $$= \frac{a^2 + 2ah + h^2 + 2a + 2h + 3 - a^2 - 2a - 3}{h} = \frac{2ah + h^2 + 2h}{h} = 2ah + h + 2$$