Math 229
06/18/2008

## Differentiation of Exponential and Logarithmic Functions

**The Derivative of the Exponential Function:** The basic rule for differentiating the exponential function is:
$\frac{d}{dx}e^x = e^x$

We'll not prove this rule, but it is in fact true that the exponential function is its own derivative!

**The Chain Rule for Exponentials:**

$\frac{d}{dx}e^{f(x)} = f'(x) \cdot e^{f(x)}$

**Examples of Derivatives Involving Exponential Functions:**

1. If $f(x) = e^{3x}$, then, using the Chain Rule for Exponentials: $f'(x) = e^{3x} \cdot (3) = 3e^{3x}$.
2. If $g(x) = e^{x^2}$, then, using the Chain Rule for Exponentials: $g'(x) = e^{x^2} \cdot (2x) = 2xe^{x^2}$.
3. If $h(x) = x^2 e^{5x}$, then, by the product rule: $h'(x) = 2xe^{5x} + 5x^2 e^{5x}$.
4. If $k(x) = \left(e^{2x} + 3x^2\right)^{\frac{5}{2}} = \frac{5}{2}\left(e^{2x} + 3x^2\right)^{\frac{3}{2}}\left(2e^{2x} + 6x\right) = \left(5e^{2x} + 15x\right)\left(e^{2x} + 3x^2\right)^{\frac{3}{2}}$
5. If $\ell(x) = e^{e^{x^2}}$, then, applying the Chain Rule for Exponentials several times: $\ell'(x) = e^{e^{x^2}} \cdot e^{x^2} \cdot x^2$

**Applications of the Derivative Involving Exponentials:**

1. Find the slope of the tangent line to $f(x) = 1 - e^{2x}$ at the point where $f$ crosses the $x$-axis. Then find the equation of the tangent line.

   First notice that if $f(x) = 0$, then $0 = 1 - e^{2x}$, so $e^{2x} = 1$, or, $\ln(e^{2x}) = \ln(1)$.
   Therefore, $2x = 0$, so $x = 0$. Therefore, the point of tangency is $(0, 0)$.
   Next, $f'(x) = -2e^{2x}$, so $m = f'(0) = -2e^0 = -2(1) = -2$.
   Thus, the tangent line has equation $y = -2x$.

2. Completely Analyze the first and second derivatives of $f(t) = te^{2t}$.

**The Derivative of Logarithmic Functions:** The basic rule for differentiating the natural logarithmic function is:

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

**Proof:**

Recall that by the inverse property of exponentials and logarithms, $e^{\ln x} = x$ for $x > 0$.
Differentiating both sides of this equation, $\frac{d}{dx}\left(e^{\ln x}\right) = \frac{d}{dx}x$
Which, by the chain rule, is: $\frac{d}{dx}(\ln x) \cdot e^{\ln x} = 1$, or, again applying the inverse property: $\frac{d}{dx}(\ln x) \cdot x = 1$.
Therefore, dividing both sides by $x$: $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

**The Chain Rule for Logarithmic Functions:**

$$\frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)}$$

**Examples of Derivatives Involving Logarithmic Functions:**

1. If $f(x) = x^2 \ln x$, then, by the product rule: $f'(x) = 2x \ln x + x^2 \cdot \frac{1}{x} = 2x \ln x + x$.
2. If $g(x) = \ln(x^2)$, then, using the Chain Rule: $g'(x) = \frac{2x}{x^2} = \frac{2}{x}$.
   Alternatively, we could have used the properties of logarithms to rewrite $g(x) = \ln(x^2)$ as $g(x) = 2\ln x$. Then $g'(x) = 2 \cdot \frac{1}{x} = \frac{2}{x}$.
3. If $h(x) = \ln[(x^2+1)(3x-2)^3]$, then, again using the properties of logarithms to rewrite $h(x)$,
   $h(x) = \ln(x^2+1) + \ln(3x-2)^3 = \ln(x^2+1) + 3\ln(3x-2)$.
   Thus $h'(x) = \frac{2x}{x^2+1} + 3 \cdot \frac{3}{3x-2} = \frac{2x}{x^2+1} + \frac{9}{3x-2}$.

**Logarithmic Differentiation:** We will skip this part of the Section in the interest of time.