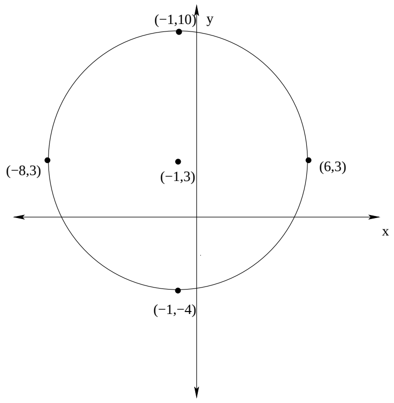
Math 261
Exam 1 - Practice Problems

1. (a) Give the equation of a circle of radius 7 centered at the point $(-1, 3)$.
   Solution:
   The general equation for a circle of radius r centered about a point (h, k) is: $(x-h)^2+(y-k)^2 = r^2$. In this case, $h = -1$, $k = 3$, and $r = 7$, so we have the equation $(x + 1)^2 + (y - 3)^2 = 49$

   (b) Sketch the graph of this circle.
   Solution:



   (c) Does this graph represent a function? Justify your answer.
   Solution:
   The graph of the circle is not the graph of a function since at fails the vertical line test at several points.

2. Given the points $A : (-2, 4)$ and $B : (7, -1)$:

   (a) Find the distance between A and B
   Solution:
   $d(A, B) = \sqrt{(7 + 2)^2 + (-1 - 4)^2} = \sqrt{9^2 + (-5)^2} = \sqrt{81 + 25} = \sqrt{106}$.

   (b) Find an equation for the line containing A and B
   Solution: First, we find the slope of the line: $m = \frac{\Delta y}{\Delta x} = \frac{4-(-1)}{-2-7} = \frac{5}{-9} = -\frac{5}{9}$.
   Next, we use the point-slope equation to find the equation for the line: $y + 1 = -\frac{5}{9}(x - 7) = -\frac{5}{9}x + \frac{35}{9}$, so $y = -\frac{5}{9}x + \frac{35}{9} - 1$, or $y = -\frac{5}{9}x + \frac{26}{9}$.

   (c) Find the equation for the line through the origin and perpendicular to $\overleftrightarrow{AB}$
   Solution: The slope of the line is the negative reciprocal of our previous line. That is, $m = \frac{9}{5}$.
   Since the line passes through the origin, $(0, 0)$, $b = 0$
   Therefore, the perpendiclar line has equation $y = \frac{9}{5}x$

3. Given the equation $4x + 3y = -2$

   (a) Find the slope of the line represented by this equation.
   Solution: Solving for y, we get $3y = -4x - 2$, or $y = -\frac{4}{3}x - \frac{2}{3}$, so the slope is $m = -\frac{4}{3}$.

   (b) Find the x and y intercepts for this line.
   Solution: From the slope-intercept form found in the work above, the y=intercept is $-\frac{2}{3}$.
   To find the x-intercept, notice that when $y = 0$ the original equation becomes $4x = -2$, so $x = -\frac{1}{2}$ is the x-intercept.

   (c) Give an equation for a line through $(-2, 4)$ that is parallel to this line.
   Since we want a line that is parallel to the original line, the slope is the same as the slope of the original line. That is, $m = -\frac{4}{3}$.
   Using the point slope formula the line we are looking for has equation:
   $y - 4 = -\frac{4}{3}(x + 2) = -\frac{4}{3}x - \frac{8}{3}$, so $y = -\frac{4}{3}x - \frac{8}{3} + 4 = -\frac{4}{3}x - \frac{8}{3} + \frac{12}{3}$
   or, in slope/intercept form: $y = -\frac{4}{3}x + \frac{4}{3}$

4. Suppose you own a company that manufactures widgets. Your supplier sells you the widgets wholesale at \$8 apiece. It costs you \$750 a month to rent your store, and you spend an additional \$2150 each month on utilities, supplies, and employee salaries. You sell the widgets at a retail price of \$15 apiece.

   (a) Find an equation $C(x)$ that gives your monthly costs, where x is the number of widgets you purchase for sale that month.

   Solution:

   The fixed monthly costs are \$2150+ \$750 = \$2900. Therefore, since the widgets cost \$8 each wholesale, the cost function is: $C(x) = 8x + 2900$.

   (b) Find equations for your monthly revenue, $R(x)$, and your monthly profit, $P(x)$, assuming that you sell all of the new widgets that you purchase.

   Solution:

   Since we charge \$15 retail for each widget, $R(x) = 15x$.
   Our profit is given by $P(x) = R(x) - C(x) = 15x - (8x + 2900) = 7x - 2900$.

   (c) How many widgets do you need to sell each month in order to break even?

   Solution:

   We break even when profit is zero, that is, when $7x - 2900 = 0$, or when $7x = 2900$, so $x \approx 414.2$. Therefore, we must sell 415 widgets to break even (well, since we rounded up, we would make a slight profit).

   (d) Suppose that instead of purchasing your widgets wholesale, you could instead build a factory and make your own widgets for only \$4 apiece. If the factory would cost \$750,000 to build, but all of your other costs would remain the same, how many widgets would you need to sell during the lifetime of the factory in order to make the construction worthwhile?

   Solution:

   The monthly cost function for our company with the new factory would be:
   $C_2(x) = 4x + 750,000 + 2900 = 4x + 752,900$. Widgets now cost us only \$4 apiece, but we have the same fixed costs, plus the cost of building the factory. It would make sense to build the facory if the factory costs us less than going with the wholesale supplier. This happens at the intersection our previous and new cost functions. That is, when $4x + 752,900 = 8x + 2900$, or when $4x = 750,000$. That is, when $x = 187,500$. Therefore, if we expect to sell over 187,500 widgets over the lifetime of the factory, it would be worth it to build the factory.

5. Simplify the following:

   (a) $\left(\frac{3}{4}\right)^{-2}$

   Solution:

   $= \left(\frac{4}{3}\right)^2 = \frac{16}{9}$

   (b) $\left(\frac{y^{12}}{25z^4}\right)^{-\frac{3}{2}}$

   Solution:

   $= \left(\frac{25z^4}{y^{12}}\right)^{\frac{3}{2}} = \left(\left(\frac{25z^4}{y^{12}}\right)^{\frac{1}{2}}\right)^3$

   $= \left(\frac{5z^2}{y^6}\right)^3 = \frac{125z^6}{y^{18}}$

   (c) $\sqrt[5]{32x^{11}y^{14}z^8}$

   Solution:

   $= 2x^2y^2z\sqrt[5]{xy^4z^3}$

   (d) $\left(\frac{(5xyz)^2z^{-2}}{2x^{-2}y^2z^{-4}}\right)^{-1}$

   Solution:

   $= \left(\frac{25x^2y^2z^2x^2z^4}{2y^2z^2}\right)^{-1} = \left(\frac{25x^4z^4}{2}\right)^{-1} = \frac{2}{25x^4z^4}$

(e) $\dfrac{3x^2-10x+3}{x^2-1} \cdot \dfrac{x^2+x-2}{x^2-9}$

Solution:

$= \dfrac{(3x-1)(x-3)}{(x+1)(x-1)} \cdot \dfrac{(x+2)(x-1)}{(x+3)(x-3)}$

$= \dfrac{(3x-1)(x+2)}{(x+1)(x+3)}$

(f) $\dfrac{2x^2+4}{2x^2+7x-4} - \dfrac{x-1}{x+4}$

Solution:

$= \dfrac{2x^2+4}{(2x-1)(x+4)} - \dfrac{x-1}{x+4} \cdot \dfrac{(2x-1)}{(2x-1)} = \dfrac{(2x^2+4)-(x-1)(2x-1)}{(2x-1)(x+4)}$

$= \dfrac{(2x^2+4)-(2x^2-3x+1)}{(2x-1)(x+4)} = \dfrac{3x+3}{(2x-1)(x+4)} = \dfrac{3(x+1)}{(2x-1)(x+4)}$

(g) $\dfrac{\frac{1}{x}+\frac{3}{x-2}}{\frac{4}{x-1}-\frac{2}{x-2}}$

Solution:

$= \dfrac{\frac{1}{x}+\frac{3}{x-2}}{\frac{4}{x-1}-\frac{2}{x-2}} \cdot \dfrac{x(x-1)(x-2)}{x(x-1)(x-2)} = \dfrac{\frac{x(x-1)(x-2)}{x}+\frac{3x(x-1)(x-2)}{x-2}}{\frac{4x(x-1)(x-2)}{x-1}-\frac{2x(x-1)(x-2)}{x-2}}$

$= \dfrac{(x-1)(x-2)+3x(x-1)}{4x(x-2)-2x(x-1)} = \dfrac{x^2-3x+2+3x^2-3x}{4x^2-8x-2x^2+2x} = \dfrac{4x^2-6x+2}{2x^2-6x}$

$= \dfrac{2(2x^2-3x+1)}{2x(x-3)} = \dfrac{2(2x-1)(x-1)}{2x(x-3)}$

(h) $\dfrac{\frac{3}{2x+2h+1}-\frac{3}{2x+1}}{h}$

Solution:

$= \dfrac{\frac{3(2x+1)}{(2x+2h+1)(2x+1)}-\frac{3(2x+2h+1)}{(2x+2h+1)(2x+1)}}{h} = \dfrac{6x+3-(6x+6h+3)}{(2x+2h+1)(2x+1)} \cdot \dfrac{1}{h}$

$= \dfrac{-6h}{(2x+2h+1)(2x+1)} \cdot \dfrac{1}{h} = \dfrac{-6}{(2x+2h+1)(2x+1)}$

6. Rationalize the denominator in the following expressions:

(a) $\dfrac{3x}{\sqrt[3]{x}}$

Solution:

$= \dfrac{3x}{\sqrt[3]{x}} \cdot \dfrac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}} = \dfrac{3x\sqrt[3]{x^2}}{x} = 3\sqrt[3]{x^2}$

(b) $\dfrac{2x+3}{\sqrt{2x}-1}$

Solution:

$= \dfrac{2x+3}{\sqrt{2x}-1} \cdot \dfrac{\sqrt{2x}+1}{\sqrt{2x}+1} = \dfrac{(2x+3)(\sqrt{2x}+1)}{2x-1}$

7. Factor each of the following expressions completely:

(a) $2x^2+x-6$

Solution:

$= (2x-3)(x+2)$

(b) $50x^2+45x-18$

Solution:

$ac = -900$, which factors as $60 \cdot (-15)$

so $50x^2+60x-15x-18 = 10x(5x+6)-3(5x+6) = (10x-3)(5x+6)$

(c) $16x^2-25y^2$

Solution:

$(4x+5y)(4x-5y)$

(d) $6x^3y - 27x^2y - 15xy$

Solution:

$3xy(2x^2 - 9x - 5) = 3xy(2x + 1)(x - 5)$

(e) $3x^3 + x^2 - 3x - 1$

$x^2(3x + 1) - (3x + 1) = (x^2 - 1)(3x + 1) = (x + 1)(x - 1)(3x + 1)$

8. Suppose that the supply and demand for a product are given by the equations $2p + 3x = 90$ and $4p - 2x = 100$, where x is the quantity sold, in thousands, and p is the price in dollars. Find the equilibrium price for this product, and the quantity sold at this price.

Solution:

Multiplying the first equation by -2, we get:

$-4p - 6x = -180$, while

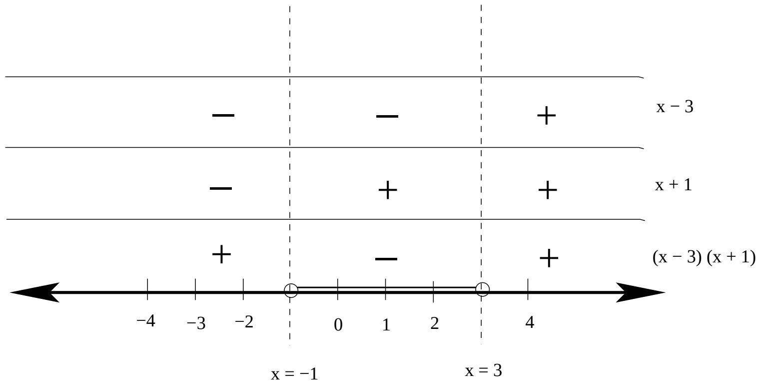
$4p - 2x = 100$

$-8x = -80$ so $x = 10$.

Plugging this into the first original equation, $2p + 30 = 90$, so $2p = 60$, so $p = 30$. Therefore the equilibrium price is \$30, and we would sell 10,000 units at this price.

9. Find the values of x that satisfy the inequality $x^2 - 2x - 3 < 0$. Graph your solution on a number line.

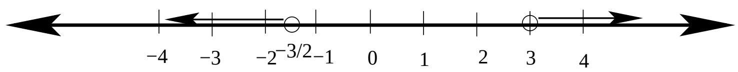
First notice that this quadratic expression factors to give $(x - 3)(x + 1) < 0$. These factors are zero at $x = 3$ and $x = -1$ respectively. We proceed using sign analysis:



10. Find the values of x that satisfy the inequality $|3 - 4x| > 9$. Graph your solution on a number line.

There are two cases: If the expression end up positive, then we have $3 - 4x > 9$, so $-4x > 6$, or $x < -\frac{3}{2}$.

On the other hand, if the expresion is negative, then $-(3 - 4x) > 9$, or $-3 + 4x > 9$, so $4x > 12$, or $x > 3$.



11. Given the function $f(x) = \dfrac{2}{x - 4}$

(a) What is the domain of f? Give your answer in interval notation.

Solution:

We need the denominator to be non-zero. Therefore, $x \neq 4$ is the domain. In interval notation, this is: $(-\infty, 4) \cup (4, \infty)$.

(b) Find $f(12)$

Solution:

$f(12) = \frac{2}{12-4} = \frac{2}{8} = \frac{1}{4}$.

(c) Find $f(2a + 4)$

Solution:

$f(2a + 4) = \frac{2}{(2a+4)-4} = \frac{2}{2a} = \frac{1}{a}$.

(d) Find $\dfrac{f(a+h)-f(a)}{h}$. Simplify your answer.

Solution:

$$\frac{\frac{2}{a+h-4}-\frac{2}{a-4}}{h}=\frac{\frac{2(a-4)-2(a+h-4)}{(a+h-4)(a-4)}}{h}=\frac{2a-8-2a-2h+8}{(a+h-4)(a-4)}\frac{1}{h}=\frac{-2h}{(a+h-4)(a-4)}\frac{1}{h}=\frac{-2}{(a+h-4)(a-4)}$$

12. Given that $f(x)=\dfrac{1}{3x-2}$ and $g(x)=\sqrt{x^2-4}$

(a) Find $\dfrac{g}{f}(x)$

Solution:

$$\frac{g}{f}(x)=\frac{\sqrt{x^2-4}}{\frac{1}{3x-2}}=(3x-2)\sqrt{x^2-4}.$$

(b) Find $g\circ f(x)$

Solution:

$$g(f(x))=\sqrt{\left(\frac{1}{3x-2}\right)^2-4}=\sqrt{\frac{1}{9x^2-12x+4}-4}$$

(c) Find $f\circ g(2)$

Solution:

$g(2)=\sqrt{2^2-4}=0$, so $f(g(2))=f(0)=\frac{1}{3(0)-2}=-\frac{1}{2}$

(d) Find the domain of $\dfrac{g}{f}$? Give your answer in interval notation.

Solution:

We need points where $g(x)$ is defined, and where $f(x)$ is defined and non-zero. For f, we must exclude the points where $3x-2=0$, so $x\neq\frac{2}{3}$. For g, we need $x^2-4\geq 0$, or $x^2\geq 4$. That is, we must have either $x\geq 2$, or $x\leq -2$. Since $x=\frac{2}{3}$ is excluded by the conditions for g, the domain of $\dfrac{g}{f}$ is given by: $(-\infty,-2]\cup[3,\infty)$.