Math 229
Exam 2 Practice Problem Solutions

1. Evaluate the following limits. Be sure to show enough work to justify your answers.

(a) $\lim_{x\to 0} \dfrac{x^2-2x}{2x^2-x-6}$

**Solution:**

$\lim_{x\to 0} \dfrac{x^2-2x}{2x^2-x-6} = \dfrac{0^2-2(0)}{2(0)^2-0-6} = \dfrac{0}{-6} = 0$

(b) $\lim_{x\to 2} \dfrac{x^2-2x}{2x^2-x-6}$

**Solution:**

First notice that this limit cannot be evaluated directly, since it leads to the indeterminate form $\frac{0}{0}$. Therefore, we try simplifying using algebra:

$\lim_{x\to 2} \dfrac{x^2-2x}{2x^2-x-6} = \lim_{x\to 2} \dfrac{x(x-2)}{(2x+3)(x-2)} = \lim_{x\to 2} \dfrac{x}{2x+3} = \dfrac{2}{2(2)+3} = \dfrac{2}{7}$

(c) $\lim_{x\to 2} \dfrac{x}{x-2}$

**Solution:**

First notice that this limit cannot be evaluated directly, since it leads to the form $\frac{0}{-2}$, which leads us to suspect that this limit might not exist. To verify this, we investigate by evaluating the expression inside the limit at points close to 2.

| $x$ | 1.9 | 2.1 | 1.99 | 2.01 | 1.999 | 2.001 |
|---|---|---|---|---|---|---|
| $f(x)$ | -19 | 21 | -199 | 201 | -1999 | 2001 |

Since the values are diverging as we get closer and closer to 2, we can conclude that the limit does not exist.

(d) $\lim_{x\to \infty} \dfrac{x^2-2x}{2x^2-x-6}$

**Solution:**

For this limit, since we are looking at the limit as it approaches positive infinity, we need only conside the highest order terms:

$\lim_{x\to \infty} \dfrac{x^2-2x}{2x^2-x-6} = \lim_{x\to \infty} \dfrac{x^2}{2x^2} = \dfrac{1}{2}$

(e) $\lim_{x\to 0} \dfrac{2x^2-x-1}{x^2-1}$

**Solution:**

$\lim_{x\to 0} \dfrac{2x^2-x-1}{x^2-1} = \dfrac{2(0)^2-(0)-1}{(0)^2-1} = \dfrac{-1}{-1} = 1$

(f) $\lim_{x\to 1} \dfrac{2x^2-x-1}{x^2-1}$

**Solution:** Here, just plugging in 1 gives us the indeterminate form $\frac{0}{0}$. To see whether or not the limit exists, we need to do further investigation. Factoring the expression inside the limit gives us

$\lim_{x\to 1} \dfrac{(2x+1)(x-1)}{(x+1)(x-1)}$, which, canceling the like terms in the fraction, gives

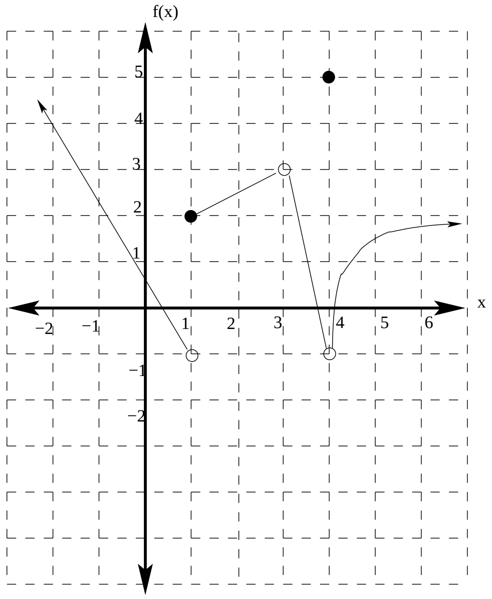
$\lim_{x\to 1} \dfrac{2x+1}{x+1} = \dfrac{2(1)+1}{1+1} = \dfrac{3}{2}.$

(g) $\lim_{x\to \infty} \dfrac{2x^2-x-1}{x^2-1}$

**Solution:** For an infinite limit involving a fraction, only the higher order terms matter. More formally, we can divide by the highest power of $x$ represented in the fraction.

$\lim_{x\to \infty} \dfrac{2x^2-x-1}{x^2-1} = \lim_{x\to \infty} \dfrac{2-\frac{1}{x}-\frac{1}{x^2}}{1-\frac{1}{x^2}} = \dfrac{2}{1} = 2$

2. Given the following graph:




(a) (3 points) Find $\lim_{x \to 1^-} f(x) = -1$

(b) (3 points) Find $\lim_{x \to 1^+} f(x) = 2$

(c) (3 points) Find $\lim_{x \to 4} f(x) = -1$

(d) (3 points) Find $\lim_{x \to \infty} f(x) = 2$

(e) (5 points) List all points where $f(x)$ is discontinuous. Explain what goes wrong at each point.

Notice that $f(x)$ is discontinuous at $x = 1, x = 3$, and $x - 4$, and is continuous everywhere else.

At $x = 1$, $f(x)$ is discontinuous since $\lim_{x \to 1} f(x)$ does not exist.

At $x = 3$, $f(x)$ is discontinuous since $f(3)$ is undefined.

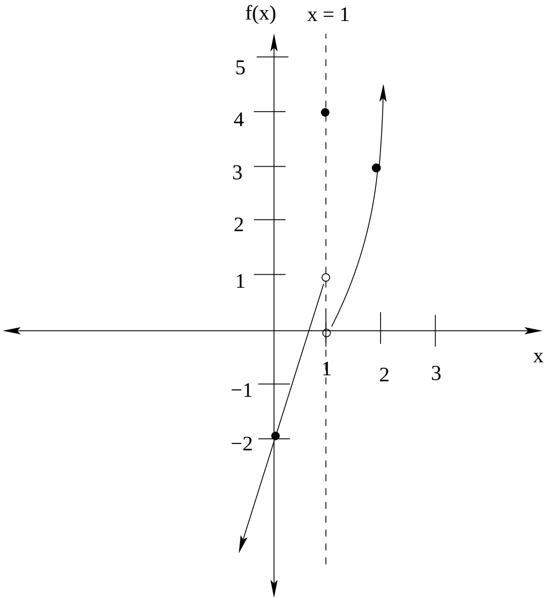
At $x = 4$, $f(x)$ is discontinuous since $\lim_{x \to 4} f(x) = -1$, while $f(4) = 5$, so the limit and the function value do not agree.

3. Given the function

$$f(x) = \begin{cases} 3x - 2 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ x^2 - 1 & \text{if } x > 1 \end{cases}$$

(a) Graph $f(x)$.

**Solution:**




(b) Find $\lim_{x \to 1} f(x)$.

**Solution:**

Notice that $\lim_{x \to 1^-} f(x) = 1$, while $\lim_{x \to 1^+} f(x) = 0$, so $\lim_{x \to 1} f(x)$ does not exist.

(c) Is $f(x)$ continuous at $x = 1$? Justify your answer.

**Solution:**

No, since the function has no limit when $x = 1$, the second condition necessary for continuity at a point is violated, so the function $f(x)$ is not continuous at $x = 1$.

4. Use the limit definition of the derivative to compute the derivative function $f'(x)$ if $f(x) = 5x^2 - 3x - 7$

**Solution:**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{5(x+h)^2 - 3(x+h) - 7 - (5x^2 - 3x - 7)}{h}$$

$$= \lim_{h \to 0} \frac{5x^2 + 10xh + 5h^2 - 3x - 3h - 7 - 5x^2 + 3x + 7}{h} = \lim_{h \to 0} \frac{10xh + 5h^2 - 3h}{h} = \lim_{h \to 0} 10x + 5h - 3$$

$$= 10x - 3$$

5. Use the limit definition of the derivative to compute the derivative function $f'(x)$ if $f(x) = 4 - 2x - 3x^2$

**Solution:**

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{4 - 2(x+h) - 3(x+h)^2 - (4 - 2x - 3x^2)}{h}$$

$$= \lim_{h \to 0} \frac{4 - 2x - 2h - 3(x^2 + 2xh + h^2) - 4 + 2x + 3x^2}{h}$$

$$= \lim_{h \to 0} \frac{4 - 2x - 2h - 3x^2 - 6xh - 3h^2 - 4 + 2x + 3x^2}{h} = \lim_{h \to 0} \frac{-2h - 6xh - 3h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(-2 - 6x - 3h)}{h} = \lim_{h \to 0} -2 - 6x - 3h = -2 - 6x.$$

6. Suppose $f(x) = x^3 - 3x^2 + 5$.

   (a) Find the equation for the tangent line to $f(x)$ when $x = 1$.
   **Solution:**
   First, we find the point of tangency by evaluating $f(x)$ when $x = 1$:
   $f(1) = 1^3 - 3(1)^2 + 5 = 1 - 3 + 5 = 3$.
   Next, we find the slope of the tangent line by evaluating the derivative of $f(x)$ when $x = 1$:
   $f'(x) = 3x^2 - 6x$, so $f'(1) = 3 - 6 = -3$.
   Finally, we use the point slope formula to find the equation for the line:
   $y - 3 = -3(x - 1) = -3x + 3$. so $y = -3x + 6$.

   (b) Find the value(s) of $x$ for which the tangent line to $f(x)$ is horizontal.
   **Solution:**
   Recall that the tangent line to a function is horizontal if and only if slope of the tangent line is zero, that is, when the derivative of the function is zero.
   Therefore, we consider the equation $f'(x) = 3x^2 - 6x = 0$, or $3x(x - 2) = 0$. This equation has two solutions: $x = 0$, and $x = 2$.

7. Suppose $f(x) = (x + 1)^{\frac{3}{2}}$.

   (a) Find the equation for the tangent line to $f(x)$ when $x = 3$.
   **Solution:**
   First notice that when $x = 3$, $f(x) = (3 + 1)^{\frac{3}{2}} = 4^{\frac{3}{2}} = 2^3 = 8$, so a the point on our curve where the tangent line meets the curve is $(3, 8)$.
   Next, to find the slope of the tangent line, we find the derivative of $f(x)$ when $x = 3$. $f'(x) = \frac{3}{2}(x + 1)^{\frac{1}{2}}$, so $f'(3) = \frac{3}{2}(3 + 1)^{\frac{1}{2}} = \frac{3}{2}(2) = 3$.
   Thus, applying the point slope formula using this information, we get
   $y - 8 = 3(x - 3)$, or $y = 3x - 1$.

   (b) Find the value(s) of $x$ for which the tangent line to $f(x)$ is horizontal.
   **Solution:**
   In part (b) above, we found that $f'(x) = \frac{3}{2}(x + 1)^{\frac{1}{2}}$. The tangent line to the original function $f(x)$ is horizontal precisely when the derivative function is zero. This occurs when $\frac{3}{2}(x + 1)^{\frac{1}{2}} = 0$, or when $(x + 1)^{\frac{1}{2}} = 0 \cdot \frac{2}{3} = 0$. That is, when $x + 1 = 0$.
   Therefore the only horizontal tangent line occurs when $x = -1$.

8. Find the derivative of each of the following functions. You **do not** have to use the limit definition, and you **do not** need to simplify your answers.

   (a) $h(x) = x^3 + \sqrt{x^3}$
   **Solution:**
   First, we rewrite $h(x) = x^3 + x^{\frac{3}{2}}$. Therefore, $f'(x) = 3x^2 + \frac{3}{2}x^{\frac{1}{2}}$

   (b) $f(x) = 5x^4 - 3x^2 + \frac{2}{x}$
   **Solution:**
   First notice that $f(x) = 5x^4 - 3x^2 + \frac{2}{x} = f(x) = 5x^4 - 3x^2 + 2x^{-1}$
   Using the Power Rule, $f'(x) = 20x^3 - 6x - 2x^{-2}$

   (c) $h(x) = \dfrac{5x^3 - 4x^2 + 7x}{x^2}$
   Here, we again rewrite $h(x)$ in order to obtain $h(x) = \frac{5x^3}{x^2} - \frac{4x^2}{x^2} + \frac{7x}{x^2} = 5x - 4 + 7x^{-1}$.
   Therefore, $h'(x) = 5 - 7x^{-2}$.

   (d) $h(x) = (x^2 - 4x^3)(4x^3 + 3x^2 - 7x + 3)$
   **Solution:**
   Using the product rule: $h'(x) = f'(x)g(x) + f(x)g'(x)$,
   $h'(x) = (2x - 12x^2)(4x^3 + 3x^2 - 7x + 3) + (x^2 - 4x^3)(12x^2 + 6x - 7)$.

(e) $f(x) = (2x^2 + 5x - 4)(x^3 + 2x^2 - 1)$
**Solution:**
Using the Product Rule, $f'(x) = (2x^2 + 5x - 4)(3x^2 + 4x) + (4x + 5)(x^3 + 2x^2 - 1)$.

(f) $f(x) = \dfrac{2x + 3}{x^2 - 1}$
**Solution:**
Using the Quotient Rule, $f'(x) = \dfrac{(x^2 - 1)(2) - (2x + 3)(2x)}{(x^2 - 1)^2}$.

(g) $h(x) = (x^3 - 2x + 1)^{\frac{5}{2}}$
**Solution:**
Using the Chain rule: $h'(x) = g'(f(x))f'(x)$, $h'(x) = \frac{5}{2}(x^3 - 2x + 1)^{\frac{3}{2}}(3x^2 - 2)$

(h) $f(x) = \sqrt{2x^2 + 1}$
**Solution:**
First notice that $f(x) = \sqrt{2x^2 + 1} = (2x^2 + 1)^{\frac{1}{2}}$
By the Chain Rule, $f'(x) = \frac{1}{2}(2x^2 + 1)^{-\frac{1}{2}}(4x) = 2x(2x^2 + 1)^{-\frac{1}{2}}$

(i) $\left(\dfrac{2 - 4x^3}{x^2 - 1}\right)^4$
**Solution:**
This derivative requires both the Chain rule and the quotient rule. If we think of $h(x) = g(f(x))$, where $g(x) = x^4$, and $f(x) = \frac{2-4x^3}{x^2-1}$, then since $f'(x) = \frac{(-12x^2)(x^2-1)-(2-4x^3)(2x)}{(x^2-1)^2}$, we see that
$h'(x) = 4\left(\dfrac{2 - 4x^3}{x^2 - 1}\right)^3 \cdot \dfrac{(-12x^2)(x^2 - 1) - (2 - 4x^3)(2x)}{(x^2 - 1)^2}$

(j) $f(x) = (x^2 + 1)(x^3 - 2x + 1)^{\frac{3}{2}}$
**Solution:**
Using the Product Rule and the Chain Rule,
$f'(x) = (x^2 + 1)\frac{3}{2}(x^3 - 2x + 1)^{\frac{1}{2}}(3x^2 - 2) + (2x)(x^3 - 2x + 1)^{\frac{3}{2}}$

9. Suppose you own a company that manufactures widgets, and the demand equation for them is given by $3x + 4p = 120$.

(a) Find the revenue function $R(x)$, and use it to compute $R(10)$ and $R(40)$.
**Solution:**
To find $R(x)$, we solve the demand equation for $p$, yielding $4p = 120 - 3x$, or $p = 30 - \frac{3}{4}x$.
Since revenue is price times quantity, $R(x) = (30 - \frac{3}{4}x)x = 30x - \frac{3}{4}x^2$.
Therefore, $R(10) = 30(10) - \left(\frac{3}{4}\right)(10)^2 = 300 - \frac{300}{4} = 300 - 75 = \$225$.
Similarly, $R(40) = 30(40) - \left(\frac{3}{4}\right)(40)^2 = 1200 - 1200 = \$0$.

(b) Find the marginal revenue function $R'(x)$
**Solution:**
$R'(x) = 30 - \frac{3}{2}x$.

(c) Compute $R'(10)$ and $R'(40)$ and explain what these numbers mean in practical terms.
**Solution:**
$R'(10) = 30 - \left(\frac{3}{2}\right)(10) = 30 - 15 = 15$. This means that when 10 units have been sold, revenue is changing at \$ 15 per widget, that is, if an additional widget were sold, revenue would increase by about \$ 15.
$R'(40) = 30 - \left(\frac{3}{2}40\right) = 30 - 60 = -30$. This means that when 40 units have been sold, revenue is changing at \$-30 per widget, that is, if an additional widget were sold, revenue would decrease by about \$ 30.

(d) If $C(x) = 20x + \frac{1}{4}x^2 + 100$, find $P(x)$ and use it to compute $P(10)$.
**Solution:**
Recall that $P(x) = R(x) - C(x) = 30x - \frac{3}{4}x^2 - (20x + \frac{1}{4}x^2 + 100) = 10x - x^2 - 100$.
Therefore, $P(10) = 10(10) - 10^2 - 100 = 100 - 100 - 100 = -100$.

(e) Find the marginal profit function $P'(x)$, use it to compute $P'(5)$, and explain what this means in practical terms.
**Solution:**
$P'(x) = 10 - 2x$
$P'(5) = 10 - 2(5) = 0$. In practical terms, this means that when 5 widgets have been sold, profit is changing at \$0 per widget. That is, profit is not changing at this point in time.

10. Suppose you own a company that manufactures snow globes, and the demand equation for them is given by $5x+4p = 200$.

(a) Find the revenue function $R(x)$, and use it to compute $R(10)$ and $R(30)$.
**Solution:**
Recall that Revenue is price times quantity sold. Solving the demand equation for $p$, we get $4p = 200 - 5x$, or $p = 50 - \frac{5}{4}x$. Therefore, $R(x) = p \cdot x = 50x - \frac{5}{4}x^2$.
$R(10) = 50(10) - \frac{5}{4}(10)^2 = 500 - \frac{5}{4}(100) = 500 - 125 = 375$
$R(30) = 50(30) - \frac{5}{4}(30)^2 = 1500 - \frac{5}{4}(900) = 1500 - 1125 = 375$

(b) Find the marginal revenue function $R'(x)$
**Solution:**
$R'(x) = 50 - \frac{5}{2}x$

(c) Compute $R'(10)$ and $R'(30)$ and explain what these numbers mean in practical terms.
**Solution:**
$R'(10) = 50 - \frac{5}{2}(10) = 50 - 25 = 25$
$R'(30) = 50 - \frac{5}{2}(30) = 50 - 75 = -25$
Notice that the marginal revenue is positive when $x = 10$ and negative when $x = 30$.
When 10 units are sold, the approximate revenue added by selling an additional unit is \$25.
When 30 units are sold, when an additional unit is sold, \$25 of revenue would be lost.