Math 229
Exam 3 Practice Problems

1. (a) Find the third derivative of $f(x)$, given that $f(x) = \dfrac{3}{x^2}$

   Solution:
   First, notice that $f(x) = \dfrac{3}{x^2} = 3x^{-2}$
   Therefore, $f'(x) = -6x^{-3}$
   $f''(x) = 18x^{-4}$
   and $f'''(x) = -72x^{-5}$

   (b) Find the second derivative of $g(x)$, given that $g(x) = \dfrac{2x+1}{3x+2}$

   Solution:
   Using the quotient rule:
   $g'(x) = \dfrac{(2)(3x+2) - (2x+1)(3)}{(3x+2)^2} = \dfrac{6x+4-6x-3}{(3x+2)^2} = \dfrac{1}{(3x+2)^2} = (3x+2)^{-2}$
   Therefore, using the chain rule, $g''(x) = -2(3x+2)^{-3}(3) = \dfrac{-6}{(3x+2)^3}$

   (c) Find the second derivative of $g(x)$, given that $h(x) = (1-x^2)^7$

   Solution:
   Using the chain rule: $h'(x) = 7(1-x^2)^6(-2x) = -14x(1-x^2)^6$.
   Then, applying the product and chain rule: $h''(x) = (-14)(1-x^2)^6 + (-14x)(6)(1-x^2)^5(-2x)$
   $= -14(1-x^2)^6 + 168x^2(1-x^2)^5$

2. Determine whether the following statements are True or False. Write a brief explanation to justify your answer.

   (a) If $f'(a) = 0$ and $f''(a) < 0$, then $(a, f(a))$ is a relative minimum of the function $f(x)$.
   False
   Since $f'(a) = 0$, $(a, f(a))$ is a critical point of $f(x)$. Also, since $f''(a) < 0$, $f(x)$ is *concave down* when $x = a$. Therefore, using the second derivative test, $(a, f(a))$ is actually a relative maximum of $f(x)$.

   (b) If $f'(x) > 0$ for $a \le x \le b$, then $(a, f(a))$ is an absolute minimum for $f(x)$ on $[a, b]$.
   True
   If $f'(x) > 0$ for $a \le x \le b$, then f is increasing on this closed interval, so the minimum value must occur at the left endpoint.

   (c) If $f(a)$ is undefined, then $x = a$ is a vertical asymptote of $f(x)$.
   False
   $f(x)$ could have a "hole" but no asymptote at this point. See problem 3(b).

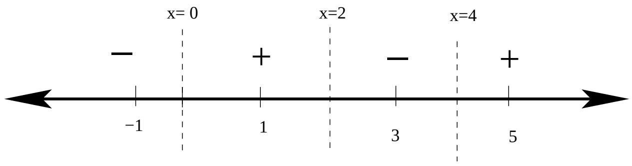
   (d) If $f(x) = \dfrac{p(x)}{q(x)}$, where both $p(x)$ and $q(x)$ are polynomials of the same degree, then $f(x)$ has a non-zero horizontal asymptote.
   True
   The horizontal asymptote will be at a height equal to the ratio of the leading terms of the polynomials.

   (e) The absolute maximum of a function $f(x)$ on an interval $[a, b]$ must occur when $x = a$, when $x = b$, or at a critical point of f inside the interval $[a, b]$.
   True
   If f is continuous, this is a consequence of the Extreme Value Theorem. Otherwise, if f is not continuous, all the points where the function is discontinuous are critical numbers since the derivative is undefined at values where a function is discontinuous.

3. Let $f(x) = x^4 - 8x^3 + 16x^2$

(a) Find the x and y intercepts of $f(x)$.
To find the y-intercept, we evaluate $f(x)$ when $x = 0$. Since $f(0) = 0$, $(0, 0)$ is the y-intercept.
To find the x-coordinates of the x-intercepts, we solve the equation $x^4 - 8x^3 + 16x^2 = 0$. Factoring this, we have $x^2(x^2 - 8x + 16) = 0$, or $x^2(x - 4)(x - 4) = 0$, so $x = 0$ and $x = 4$ are the solutions to this equation. Thus, $(0, 0)$ and $(4, 0)$ are the x-intercepts of $f(x)$.

(b) Find the intervals where $f(x)$ is increasing and the intervals where $f(x)$ is decreasing.
First notice that $f'(x) = 4x^3 - 24x^2 + 32x = 4x(x^2 - 6x + 8) = 4x(x - 4)(x - 2)$.
Therefore, the critical points occur when $x = 0$, $x = 2$, and $x = 4$.
Using sign analysis:



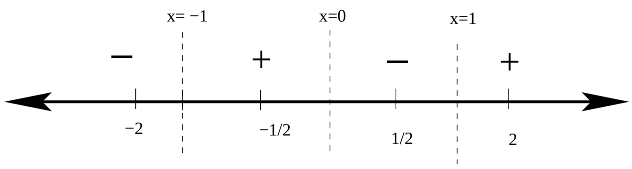
Thus $f(x)$ is increasing on: $(0, 2) \cup (4, \infty)$
And $f(x)$ is decreasing on: $(-\infty, 0) \cup (2, 4)$

(c) Find and classify the relative extrema of $f(x)$.
First notice that $f(0) = 0$, $f(2) = 2^4 - 8(2)^3 + 16(2)^2 = 16 - 64 + 64 = 16$, and $f(4) = 4^4 - 8(4)^3 + 16(4)^2 = 256 - 512 + 256 = 0$.
From our sign analysis above: $(0, 0)$ and $(4, 0)$ are relative minima, while $(2, 16)$ is a relative maximum.

(d) Find the equation of the tangent line to $f(x)$ when $x = 1$.
To find the tangent line, we need a point and a slope. To find the point, we evaluate $f(1) = 1 - 8 + 16 = 9$, giving us the point $P = (1, 9)$. To find the slope, we evaluate $f'(1) = 4 - 24 + 32 = 12$, giving us the slope $m = 12$. Therefore, using the point slope formula, $y - 9 = 12(x - 1)$, or $y = 12x - 3$.

4. Given that $f(x) = \frac{1}{5}x^5 - \frac{2}{3}x^3$:

(a) Find the intervals where $f(x)$ is concave up and the intervals where $f(x)$ is concave down.
First, notice that $f''(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x + 1)(x - 1)$. Therefore, the key values of $f''(x)$ occur when $4x(x + 1)(x - 1) = 0$, or at $x = 0$, $x = -1$, and $x = 1$.
Using sign analysis:



Thus $f(x)$ is concave up on: $(-1, 0) \cup (1, \infty)$
And $f(x)$ is concave down on: $(-\infty, -1) \cup (0, 1)$

(b) Find the coordinates of the inflection points of $f(x)$.
Notice that $f(0) = 0$, $f(1) = \frac{1}{5} - \frac{2}{3} = \frac{3}{15} - \frac{10}{15} = -\frac{7}{15}$, and $f(-1) = -\frac{1}{5} + \frac{2}{3} = -\frac{3}{15} + \frac{10}{15} = \frac{7}{15}$
Therefore, the inflection points are: $(0, 0)$, $(1, -\frac{7}{15})$, and $(-1, \frac{7}{15})$.

5. Find the horizontal and vertical asymptotes (if any) of the following functions. (You do **not** need to sketch the graphs)

(a) $f(x) = \dfrac{2x - 3}{x^2 - 1}$.
First notice that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{2x - 3}{x^2 - 1} = \lim_{x\to\infty} \dfrac{2x}{x^2} = \lim_{x\to\infty} \dfrac{2}{x} = 0$, so $f(x)$ has a horizontal asymptote $y = 0$.
Next, notice that the fraction is undefined when $x^2 - 1 = 0$, or when $x = \pm 1$. Since neither of these values make the numerator $2x - 3$ zero, $f(x)$ has two vertical asymptotes: $x = 1$ and $x = -1$.

(b) $f(x) = \dfrac{3x^2 - 3x}{x^2 - 1}$.

First notice that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{3x^2 - 3x}{x^2 - 1} = \lim_{x\to\infty} \dfrac{3x^2}{x^2} = \lim_{x\to\infty} \dfrac{3}{1} = 3$, so $f(x)$ has a horizontal asymptote $y = 3$. Next, notice that the fraction is undefined when $x^2 - 1 = 0$, or when $x = \pm 1$. These are both potential asymptotes. However, notice that when at $x = 1$, $3x^2 - 3x = 3 - 3 = 0$, so $f(x)$ has a hole but not an asymptote at this point. On the other hand, $3x^2 - 3x = 3 + 3 = 6$ when $x = -1$, which is non-zero, so $f(x)$ has one vertical asymptote: $x = -1$.

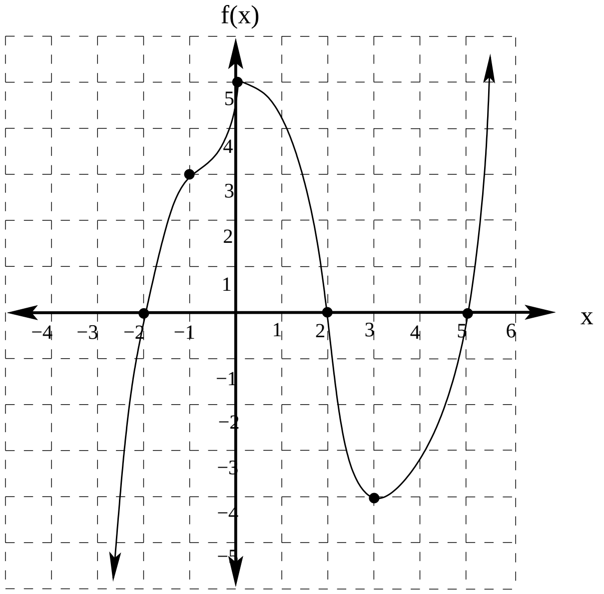
6. Carefully draw the graph of a function satisfying the following conditions:

x-intercepts: $(-2, 0), (2, 0), (5, 0)$; y-intercept: $(0, 5)$

Increasing on $(-\infty, 0) \cup (3, \infty)$ and Decreasing on $(0, 3)$

Concave Up on $(-1, 0) \cup (2, \infty)$ and Concave Down on $(-\infty, -1) \cup (0, 2)$

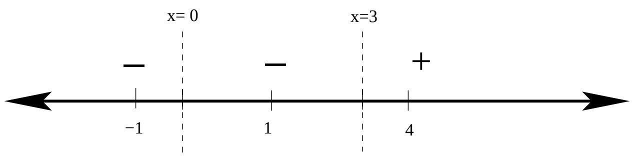
$f(-1) = 3$, and $f(3) = -4$.



7. Let $f(x) = \frac{1}{4}x^4 - x^3$

(a) Find the x and y intercepts of $f(x)$.
y-intercept: When $x = 0$, $f(x) = \frac{1}{4}0^4 - 0^3 = 0$, so $(0, 0)$ is the y-intercept.
x-intercepts: If $f(x) = \frac{1}{4}x^4 - x^3 = 0$, then $x^3(\frac{1}{4}x - 1) = 0$, so either $x^3 = 0$, so $x = 0$, or $\frac{1}{4}x - 1 = 0$, so $\frac{1}{4}x = 1$, thus $x = 4$, so $(0, 0)$ and $(4, 0)$ are the x-intercepts.

(b) Find the intervals where $f(x)$ is increasing and those where $f(x)$ is decreasing.
Since $f'(x) = x^3 - 3x^2$, the critical points occur when $x^3 - 3x^2 = 0$, or when $x^2(x - 3) = 0$, therefore, the critical values are $x = 0$ and $x = 3$. We do sign analysis using the test points $x = -1$, $x = 1$, and $x = 4$ and observe that $f'(-1) = -1 - 3 = -4$, $f'(1) = 1 - 3 = -2$, and $f'(4) = 64 - 48 = 16$.



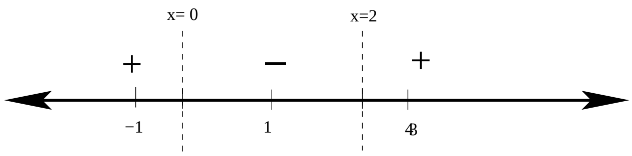
Therefore, $f(x)$ is increasing on $(3, \infty)$ and decreasing on $(-\infty, 0) \cup (0, 3)$.

(c) Find and classify the relative extrema of $f(x)$.

Since the first derivative does not change signs when we cross $x = 0$, there is not an extremum there. On the other hand, the derivative goes from negative to positive when we move across $x = 3$, so the point $(3, f(3)) = (3, -\frac{27}{4})$ is a relative minimum.

(d) Find the intervals where $f(x)$ is concave up and those where $f(x)$ is concave down.

Since $f''(x) = 3x^2 - 6x$, the critical points occur when $3x^2 - 6x = 0$, or when $3x(x - 2) = 0$, therefore, the critical values are $x = 0$ and $x = 2$. We do sign analysis using the test points $x = -1$, $x = 1$, and $x = 3$ and observe that $f''(-1) = 3 + 6 = 9$, $f''(1) = 3 - 6 = -3$, and $f''(3) = 27 - 18 = 9$.



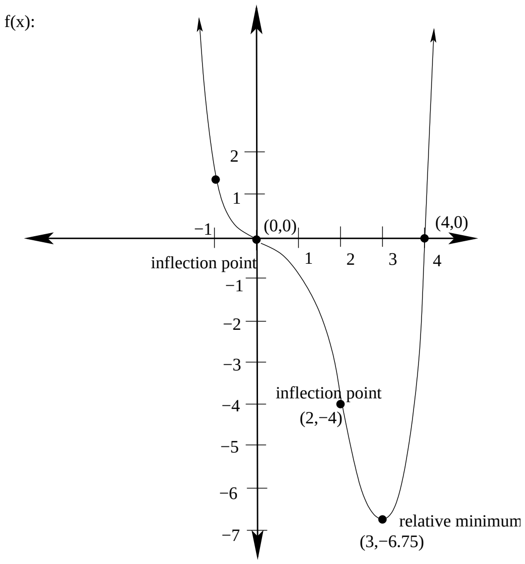
Therefore, $f(x)$ is concave down on $(0, 2)$ and concave up on $(-\infty, 0) \cup (2, \infty)$.

(e) Find any inflection points of $f(x)$.

Notice that the second changes signs both when $x = 0$ is crossed and when $x = 2$ is crossed. Therefore, both $(0, 0)$, and $(2, f(2)) = (2, -4)$ are inflection points.

(f) Graph $f(x)$. Be sure to label all relative extrema, intercepts, and inflection points.

Putting all these facts together yields the following graph:



8. Suppose the daily cost for producing x widgets is given by $C(x) = 5x^2 - 20x + 500$, where $C(x)$ is in dollars, and a maximum of 20 widgets can be produced each day.

(a) Find the production level which minimizes the daily costs. Also find the daily cost at this production level.

We are looking for the absolute minimum of $C(x)$ on the interval $[0, 20]$. To find this, we first compute $C'(x) = 10x - 20$, and notice that it has a single critical point when $10x - 20 = 0$, or when $x = 2$.

Since the absolute extrema of a function on an interval must occur at either a critical point or one of the endpoints, we check to see what the cost is at each of these points of interest:

$C(0) = 500$, $C(2) = 5(4) - 20(2) + 500 = 20 - 40 + 500 = 480$, and $C(20) = 5(400) - 20(20) + 500 = 2000 - 400 + 500 = 2100$.

Therefore, the cost is minimized when 2 widgets are produced at a cost of \$480.

(b) Find the production level which minimizes the **average** cost per widget. Also find the average cost per widget at this production level.

Here, we are interested in minimizing the average cost per widget produced rather than the total cost of production. To do this, we want to find the absolute minimum of the average cost function $\overline{C}(x) = \frac{C(x)}{x} = \frac{5x^2-20x+500}{x} = 5x - 20 + 500x^{-1}$ on the interval $[0, 20]$.

We use the same procedure as above:

$\overline{C}'(x) = 5 - 500x^{-2} = 5 - \frac{500}{x^2} = \frac{5x^2-500}{x^2}$, which has critical points of both types. $x = 0$ makes the derivative undefined, and if $5x^2 - 500 = 0$, then $5x^2 = 500$, so $x^2 = 100$, or $x = \pm 10$. Notice that $x = -10$ is not in our interval of interest.

Since the absolute extrema of a function on an interval must occur at either a critical point or one of the endpoints, we check to see what the average cost is at each of these points of interest:

$\overline{C}(0)$ is undefined (division by zero).

$\overline{C}(10) = 5(10) - 20 + \frac{500}{10} = 50 - 20 + 50 = 80$

$\overline{C}(20) = 5(20) - 20 + \frac{500}{20} = 100 - 20 + 25 = 105$

Therefore, the absolute minimum average cost of \$80 per widget occurs when 10 widgets are produced.

9. Suppose the daily cost and revenue for producing x widgets are given by the functions: $C(x) = 750 - 3x + .005x^2$ and $R(x) = 825 + 2x - .005x^2$ for $0 \leq x \leq 400$. Find the production level which maximizes daily **profits**. Also find the amount of profit at this production level.

Recall that $P(x) = R(x) - C(x) = (825 + 2x - .005x^2) - (750 - 3x + .005x^2) = 75 + 5x - .01x^2$

Therefore, $P'(x) = 5 - .02x$, which is zero when $.02x = 5$, or when $x = 250$.

We check the profit level at both the endpoints and the critical point in order to find the absolute maximum profit.

$P(0) = 75 + 5(0) - .01(0)^2 = 75$

$P(250) = 75 + 5(250) - .01(250)^2 = 75 + 1250 - 625 = 700$

$P(400) = 75 + 5(400) - .01(400)^2 = 75 + 2000 - 1600 = 475$

Thus profit is maximized at \$700 when 250 widgets are produced.

10. (16 points) Suppose you own a furniture store and sign a contract with a retailer to supply chairs. The terms of the contract state that you will charge \$90 per chair when up to 300 chairs are ordered, but you will reduce the price by 25 cents per chair (on the entire order) for every chair ordered over 300, up to a total of 100 additional chairs. What is the largest revenue you can make under this contract? [Hint: Consider the cases $0 \leq x \leq 300$ and $300 < x \leq 400$ separately]

Solution:

First, when $0 \leq x \leq 300$, $R(x) = 90x$, where x is the number of chairs sold. Since $R'(x) = 90$, in this case, we (not surprisingly) have no critical points, so we need only check the boundary points:

$R(0) = 0$, and $R(300) = 30 \cdot 90 = \$27{,}000$.

When $300 < x \leq 400$, the price depends on the number of chairs sold (specifically, on $x - 300$, the number of chairs in excess of 300 that are sold, since we drop the price on the entire order by 25 cents for each chair sold in excess of 300). Therefore, $R(x) = (\text{price})\cdot(\text{quantity}) = (90 - .25(x - 300)) \cdot (x) = (90 - .25x + 75)x = 165x - .25x^2$.

Then $R'(x) = 165 - .5x$, which equals zero when $.5x = 165$, or when $x = 330$. We then check to see what the revenue is at this critical point, as well as on the boundary point.

$R(330) = 165(330) - .25(330)^2 = 27,225$

$R(400) = 165(400) - .25(400)^2 = 26,000$

Therefore, our revenue attains an absolute maximum of \$27,225 when 330 chairs are sold.