Math 229

Extrema of Functions

The Extreme Value Theorem:

If a function f is continuous on a closed interval $[a, b]$, then f takes on a minimum and maximum value at least once in $[a, b]$.

Note: If we consider an f that is not continuous or f defined on a set that is not a closed interval, then f may or may not attain a minimum and maximum value.

Theorem 1 If a function f has a local extrema at a number c in an open interval, then either $f'(c) = 0$ or $f'(c)$ does not exist.

Corollary If $f'(c)$ exists and $f'(c) \neq 0$, then $f(c)$ is *not* a local extrema of the function f.

Theorem 2 If a function f is continuous on a closed interval $[a, b]$ and has its maximum or minimum value at a number c in the open interval, (a, b), then either $f'(c) = 0$ or $f'(c)$ does not exist.

Definition A number c in the domain of a function f is a *critical number* of f if either $f'(c) = 0$ or $f'(c)$ does not exist.

Method for Finding the Extrema of a Continuous Function on a Closed Interval:

Suppose f is a function that is continuous on the closed interval $[a, b]$. From the Extreme Value Theorem, we know that it has both maximum and minimum values in the interval $[a, b]$. From Theorem 4.7, we know that if the max and min values occur in the open interval (a, b), then they must occur at a critical number. Therefore, we can find the max and min values of f by checking the values of our function at all critical points and at the end points:

- Differentiate f, and find all critical numbers c of f in (a, b).
- Compute $f(c)$ for each critical number in (a, b).
- Compute $f(a)$ and $f(b)$, the values of the function on the two endpoints.
- Compare values: the biggest value is the max, the smallest in the min.

Example: Consider $f(x) = x^4 - 2x^2 + 17$ on the interval $[-2, 2]$

Notice that $f'(x) = 4x^3 - 4x$, and this function is defined everywhere, so the critical numbers of f all occur when $f'(x) = 0$. That is, when $4x^3 - 4x = 0$, or $4x(x^2 - 1) = 4x(x + 1)(x - 1) = 0$.

Then the critical numbers are $c = 0, -1$, and 1.

Notice $f(0) = 17$, $f(-1) = 1 - 2 + 17 = 16$, and $f(1) = 1 - 2 + 17 = 16$.

Checking the endpoints, $f(-2) = (-2)^4 - 2(-2)^2 + 17 = 16 - 8 + 17 = 25$ and $f(2) = (2)^4 - 2(2)^2 + 17 = 16 - 8 + 17 = 25$

Thus, the maximum value of f on the interval $[-2, 2]$ is 25, and the minimum value of f on the interval $[-2, 2]$ is 16.