Math 127
Exam 2 Practice Problem Solutions

1. True or False:

   (a) Every quadratic equation has two distinct real solutions.
   Solution: False. For example, $x^2 + 2x + 1 = 0$ has only one solution $(x^2 + 2x + 1 = (x + 1)^2)$

   (b) Every absolute value equation has two distinct real solutions.
   Solution: False. For example, consider $|2x - 4| = -1$ and $|2x - 4| = 0$, which have no solution and 1 solution, respectively.

   (c) $\sqrt{i^4} = -1$
   Solution: False. Since $i^4 = 1$, $\sqrt{i^4} = 1$, not -1. If you tried to simplify by removing i^2 from under the radical and got -1, you forgot that you must take the absolute value of expressions removed from under an even root.

   (d) Every inequality has infinitely many solutions.
   Solution: False. For example, $x^2 \leq 0$ has only one solution, and $x^2 < 0$ has no solutions.

   (e) $x = 0$ is a solution to the equation $x(x - 2) = 4$
   Solution: False. Notice that $0(0 - 2) = (0)(-2) = 0 \neq 4$.

2. Use completing the square to solve the quadratic equation $3x^2 - 12x + 5 = 0$.
   Solution:
   $3x^2 - 12x = -5$
   $x^2 - 4x = -\frac{5}{3}$
   $x^2 - 4x + \left(-\frac{4}{2}\right)^2 = -\frac{5}{3} + \left(-\frac{4}{2}\right)^2$
   $x^2 - 4x + 4 = -\frac{5}{3} + 4 = -\frac{5}{3} + \frac{12}{3} = \frac{7}{3}$
   $(x - 2)^2 = \frac{7}{3}$
   $x - 2 = \pm\frac{\sqrt{7}}{\sqrt{3}}$
   $x = 2 \pm \frac{\sqrt{21}}{3}$

3. Solve the following quadratic equations:

   (a) $4x^2 - 5x + 10 = 2x^2 - 8x + 12$
   Solution:
   Moving all the terms to the same side, we get $2x^2 + 3x - 2 = 0$.
   Factoring, we then have $(2x - 1)(x + 2) = 0$
   Therefore, we either have $(2x - 1) = 0$, in which case, $2x = 1$, so $x = \frac{1}{2}$, or $x + 2 = 0$, so $x = -2$.

   (b) $3x^2 + 10 = 5x$
   Solution:
   Moving all the terms to one side, we get $3x^2 - 5x + 10 = 0$. This does not factor, so we proceed by using the quadratic formula with $a = 3, b = -5$, and $c = 10$.
   Therefore, $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{5\pm\sqrt{25-(4)(3)(10)}}{(2)(3)} = \frac{5\pm\sqrt{25-120}}{6} = \frac{5\pm\sqrt{-95}}{6} = \frac{5}{6} \pm \frac{\sqrt{(-1)(19)(5)}}{6} = \frac{5}{6} \pm \frac{i\sqrt{95}}{6}$.

4. Perform the indicated operations and express your answer in the form $a + bi$:

   (a) $(3 - 2i) - (12 + 6i)$
   Solution: $(3 - 2i) - (12 + 6i) = (3 - 12) + (-2i - 6i) = -9 - 8i$

   (b) $(3 - 2i)(5 + 3i)$
   Solution: $(3 - 2i)(5 + 3i) = 15 - 10i + 9i - 6i^2 = 15 + 6 - i = 21 - i$.

   (c) $\dfrac{3 - 2i}{3i}$
   Solution: $\dfrac{3 - 2i}{3i} = \dfrac{(3 - 2i) \cdot i}{(3i) \cdot i} = \dfrac{3i - 2i^2}{3i^2} = \dfrac{3^i + 2}{-3} = -\dfrac{2}{3} - i$

(d) $\dfrac{3-2i}{2+3i}$

Solution: $\dfrac{3-2i}{2+3i} = \dfrac{(3-2i)(2-3i)}{(2+3i)(2-3i)} = \dfrac{6-4i-9i+6i^2}{4+6i-6i-9i^2} = \dfrac{6-13i+6i^2}{4-9i^2} = \dfrac{6-13i-6}{4+9}$

$= \dfrac{-13i}{13} = -i$

(e) i^{3147}

Solution: Notice $3147 = 4(786)+3$, so $i^{3147} = (i^4)^{786} i^3 = 1^{786} i^3 = i^3 = -i$.

5. Solve the following equations:

(a) $7x + 2 = -12$

Solution: $7x = -14$, so $x = -2$

(b) $4(x-1) + 3(2-x) = 10$

Solution: $4x - 4 + 6 - 3x = 10$, or $x + 2 = 10$

Therefore, $x = 8$

(c) $\dfrac{3}{10}x - \dfrac{3}{5} = \dfrac{3}{2}$

Solution: Multiplying to clear the denominators, $10 \cdot \left[\dfrac{3}{10}x - \dfrac{3}{5}\right] = 10 \cdot \left[\dfrac{3}{2}\right]$

$3x - 6 = 15$, or $3x = 21$. Hence $x = 7$

(d) $-20x = 5x^2$

Solution: Collecting all terms on one side, $5x^2 + 20x = 0$.

Factoring, $5x(x+4) = 0$, so either $5x = 0$, or $x + 4 = 0$

Thus $x = 0$, or $x = -4$

(e) $(x+6)(x-2) = -7$

Solution: Notice that we must one again collect all terms on one side.

Multiplying, $x^2 + 6x - 2x - 12 = -7$, so, combining terms, $x^2 + 4x - 5 = 0$

Factoring, $(x+5)(x-1) = 0$, so either $x + 5 = 0$ or $x - 1 = 0$

Thus $x = -5$, or $x = 1$

(f) $\dfrac{3}{x+6} - \dfrac{1}{x-2} = \dfrac{-6}{x^2+4x-12}$

Solution: Factoring, $\dfrac{3}{x+6} - \dfrac{1}{x-2} = \dfrac{-6}{(x+6)(x-2)}$

Multiplying to clear the denominators, $(x+6)(x-2) \cdot \left[\dfrac{3}{x+6} - \dfrac{1}{x-2}\right] = (x+6)(x-2) \cdot \left[\dfrac{-6}{(x+6)(x-2)}\right]$

$= 3(x-2) - (x+6) = -6$, or $3x - 6 - x - 6 + 6 = 0$.

Then $2x - 6 = 0$, or $2x = 6$. Therefore, $x = 3$.

(g) $|2 - 3x| - 3 = 5$

Note that it is a good idea to isolate the absolute value expression before splitting into two cases.

$|2 - 3x| = 8$, so we have two possible cases:

$2 - 3x = 8$ or $2 - 3x = -8$

Then $-3x = 6$ or $-3x = -10$

So either $x = -2$ or $x = \frac{10}{3}$

(h) $x^4 - x^3 - 9x^2 + 9x = 0$

Solution: Notice that $x^4 - x^3 - 9x^2 + 9x = x(x^3 - x^2 - 9x + 9) = x[x^2(x-1) - 9(x-1)]$

$= x(x^2 - 9)(x-1) = x(x+3)(x-3)(x-1) = 0$.

Therefore we must have $x = 0$, $x + 3 = 0$, $x - 3 = 0$, or $x - 1 = 0$

Hence $x = 0, x = -3, x = 3$, and $x = 1$ are the solutions to this equation.

(i) $8x - x^{\frac{5}{3}} = 0$

Solution: $8x - x^{\frac{5}{3}} = x(8 - x^{\frac{2}{3}}) = 0$, so either $x = 0$ or $8 - x^{\frac{2}{3}} = 0$.

If $8 - x^{\frac{2}{3}} = 0$, then $8 = x^{\frac{2}{3}}$, so $(8)^{\frac{3}{2}} = \left(x^{\frac{2}{3}}\right)^{\frac{3}{2}}$. So $x = (8)^{\frac{3}{2}} = \sqrt{8^3} = 8\sqrt{8} = 16\sqrt{2}$.

Hence, our solutions are $x = 0$ or $x = 16\sqrt{2}$. (Notice that both of these solutions check: $8(0) - 0^{\frac{5}{3}} = 0$, and $8(16\sqrt{2}) - (16\sqrt{2})^{\frac{5}{3}} = 8 \cdot (8)^{\frac{3}{2}} - ((8)^{\frac{3}{2}})^{\frac{5}{3}} = 8^{\frac{5}{2}} - 8^{\frac{5}{2}} = 0$).

(j) $2 - \sqrt[3]{2x + x^2} = 0$

Solution:

Isolating the radical, we obtain $2 = \sqrt[3]{2x + x^2}$.

Cubing both sides, we then have $2^3 = \left(\sqrt[3]{2x + x^2}\right)^3$, or $8 = 2x + x^2$.

Moving everything to one side gives $x^2 + 2x - 8 = 0$, which factors as $(x + 4)(x - 2) = 0$.

Therefore, $x = -4$ and $x = 2$ are our potential solutions.

Checking these: $\sqrt[3]{2(-4) + (-4)^2} = \sqrt[3]{-8 + 16} = \sqrt[3]{8} = 2$, and $2 - 2 = 0$,

while $\sqrt[3]{2(2) + 2^2} = \sqrt[3]{4 + 4} = \sqrt[3]{8} = 2$, and $2 - 2 = 0$.

(k) $x + 5 = \sqrt{2x + 13}$

Solution: Squaring both sides: $(x + 5)^2 = 2x + 13$, or $x^2 + 10x + 25 = 2x + 13$

Therefore, $x^2 + 10x + 12 = 0$. Factoring this, $(x + 6)(x + 2) = 0$.

So either $x + 6 = 0$ or $x + 2 = 0$. Hence $x = -6$ or $x = -2$

Now, since we squared both sides to solve this equation, we **must** check our solutions.

If $x = -6$, then the left hand side of the original equation yields $-6 + 5 = -1$ while the right side yields $\sqrt{2(-6) + 13} = \sqrt{1} = 1$, so this solution does not check.

If $x = -2$, then the left hand side of the original equation yields $-2 + 5 = 3$ while the right side yields $\sqrt{2(-2) + 13} = \sqrt{9} = 3$, so this solution does check.

(l) $\sqrt{x + 8} = 2 + \sqrt{x}$

Solution: Squaring both sides: $x + 8 = (2 + \sqrt{x})$, or $x + 8 = 4 + 4\sqrt{x} + x$

Isolating the remaining radical term, $4 = 4\sqrt{x}$, or $1 = \sqrt{x}$.

Squaring again, $1 = x$.

Checking this solution: $\sqrt{1 + 8} = \sqrt{9} = 3$, while $2 + \sqrt{1} = 2 + 1 = 3$, so this solution is valid.

(m) $(y + 3)^{\frac{2}{3}} - 2(y + 3)^{\frac{1}{3}} - 3 = 0$

Solution: Here, we have an equation that is quadratic in form. By substituting $u = (y + 3)^{\frac{1}{3}}$, we obtain $u^2 - 2u - 3 = 0$, which factors as $(u - 3)(u + 1) = 0$.

Therefore, we have $u = 3$, or $u = -1$.

Going back to our substitution equation, if $(y + 3)^{\frac{1}{3}} = 3$, then, cubing both sides, $y + 3 = 27$, or $y = 24$.

On the other hand, if $(y + 3)^{\frac{1}{3}} = -1$, then, cubing both sides, $y + 3 = -1$, or $y = -4$.

Checking these, $(24 + 3)^{\frac{2}{3}} - 2(24 + 3)^{\frac{1}{3}} - 3 = (27)^{\frac{2}{3}} - 2(27)^{\frac{1}{3}} - 3 = 3^2 - 2(3) - 3 = 9 - 6 - 3 = 0$,

while $(-4 + 3)^{\frac{2}{3}} - 2(-4 + 3)^{\frac{1}{3}} - 3 = (-1)^{\frac{2}{3}} - 2(-1)^{\frac{1}{3}} - 3 == (-1)^2 - 2(-1) - 3 = 1 + 2 - 3 = 0$.

6. Solve the following inequalities. Express your answer in interval notation.

(a) $15 \leq -5x$

Solution: Dividing by -5 and reversing the inequality, $-3 \geq x$.

In interval notation this is: $(-\infty, -3]$

(b) $.3x - .2(3x + 1) < 1$

Solution: Distributing, $.3x - .6x - .2 < 1$, or $-.3x < 1.2$

Multiplying by 10, $-3x < 12$. Then, dividing by -3, $x > -4$.

In interval notation this is: $(-4, \infty)$

(c) $-7 < 3x - 4 \leq 5$

Solution: Adding 4 to each term, $-3 < 3x \leq 9$

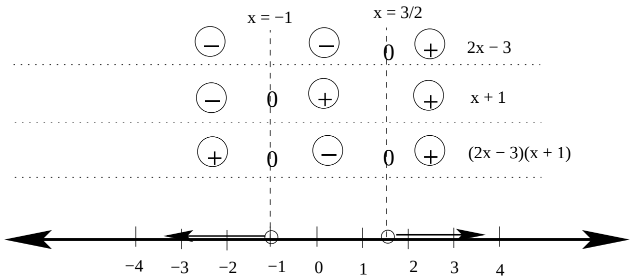
Then, dividing by 3, $-1 < x \leq 3$, which in interval notation this is: $(-1, 3]$

(d) $2x - 5 \leq 5x - 2 \leq 2x + 7$
Solution: Subtracting $2x$ from each part of the inequality, we get $-5 \leq 3x - 2 \leq 7$.
Adding 2 to each part, we then have $-3 \leq 3x \leq 9$. Dividing by 3, we then have $-1 \leq x \leq 3$, which, in interval notation, is expressed $[-1, 3]$.

(e) $|5x - 1| > 4$
Solution: Splitting into two cases.
Positive case: $5x - 1 > 4$ Negative case: $-(5x - 1) > 4$ or $5x - 1 < -4$
Then $5x > 5$ or $5x < -3$
Thus $x > 1$ or $x < \frac{-3}{5}$
In interval notation, this is $(-\infty, \frac{-3}{5}) \cup (1, \infty)$

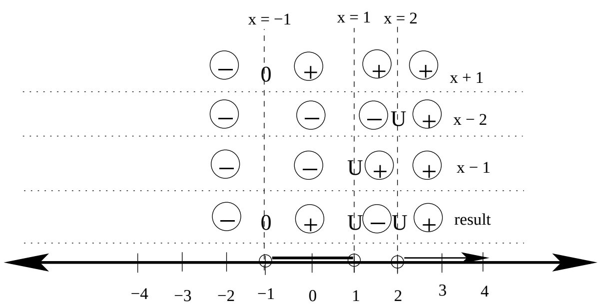
(f) $|2x - 3| + 6 \leq 11$
Solution: First, we isolate the absolute value expression $|2x - 3| \leq 5$. Then we split into two cases.
Positive case: $2x - 3 \leq 5$ Negative case: $-(2x - 3) \leq 5$ or $2x - 3 \geq -5$
Then $2x \leq 8$ or $2x \geq -2$
Thus $x \leq 4$ and $x \geq -1$
In interval notation, this is $[-1, 4]$

(g) $2x^2 - x - 3 > 0$
Solution: Factoring, we have $(2x - 3)(x + 1) > 0$. These have key values given by $2x - 3 = 0$, so $2x = 3$, or $x = \frac{3}{2}$, and $x + 1 = 0$, so $x = -1$. We proceed by doing factor analysis:



Then, our solution in interval notation is: $(-\infty, -1) \cup (\frac{3}{2}, \infty)$.

(h) $\dfrac{3}{x-2} > \dfrac{2}{x-1}$
Solution: Our strategy is to get everything to one side, and then to combine the interesting side into a single fraction.
$\dfrac{3}{x-2} - \dfrac{2}{x-1} > 0$, so $\dfrac{3(x-1)}{(x-2)(x-1)} - \dfrac{2(x-2)}{(x-1)(x-2)} > 0$
But then $\dfrac{(3x-3)-(2x-4)}{(x-1)(x-2)} > 0$, or $\dfrac{(x+1)}{(x-1)(x-2)} > 0$.
This gives a fraction with 3 terms whose key values are $x = -1$, $x = 1$, and $x = 2$. We again proceed by doing factor analysis:



Our solutions, in interval notation, are: $(-1, 1) \cup (2, \infty)$

7. Use algebra to solve each of the following. You must clearly define your variables and state your conclusion in a sentence.

(a) One number is 4 times another. The sum of their reciprocals is $\frac{1}{4}$. Find the numbers.

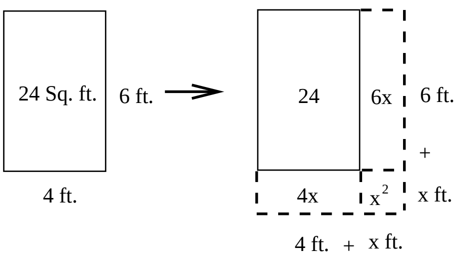
Solution: Let x be one number and $4x$ the other. Then an expression for the sum of their reciprocals is:
$\frac{1}{x} + \frac{1}{4x} = \frac{1}{4}$
Multiplying to clear denominators, $4x \cdot \left[\frac{1}{x} + \frac{1}{4x}\right] = 4x \cdot \left[\frac{1}{4}\right]$
which gives $\frac{4x}{x} + \frac{4x}{4x} = \frac{4x}{4}$, or $4 + 1 = x$, so $x = 5$
Check: $\frac{1}{5} + \frac{1}{4(5)} = \frac{1}{5} + \frac{1}{20} = \frac{4}{20} + \frac{1}{20} = \frac{5}{20} = \frac{1}{5}$

(b) A boat travels 24 miles upstream in the same time it takes to travel 30 miles downstream. The current is 2 mph. Find the speed of the boat in still water.

Solution: Let r be the speed of the boat in still water. Since the current is 2 mph, then the speed of the boat upstream is $r - 2$ mph, and $r + 2$
Now, since the time taken for each trip is the same, we let t be the travel time for each half.
Then the trip upstream is modeled by: $24 = (r - 2)t$ and the trip downstream is modeled by: $30 = (r + 2)t$
Solving for t, $t = \frac{24}{r-2}$ and $t = \frac{30}{r+2}$.
Therefore, we have the equation: $\frac{24}{r-2} = \frac{30}{r+2}$
Multiplying by $(r - 2)(r + 2)$ to clear our denominators yields: $24(r + 2) = 30(r - 2)$
So $24r + 48 = 30r - 60$, the hence $108 = 6r$, or $r = 18$
That is, the boat travels 18 miles per hour in still water.

(c) Tony has a rectangular garden that measures 4 feet by 6 feet. He wants to double its area by increasing the garden's length and width by the same amount. Find the length and width of his new garden.

Solution:



If we let x be the amount added to the dimensions of the garden, then we have $24 \cdot 2 = (4 + x)(6 + x)$, or $48 = 24 + 6x + 4x + x^2$. Moving everything over to one side, we have $x^2 + 10x - 24 = 0$, which factors to give $(x + 12)(x - 2) = 0$. Our two potential solutions are $x = -12$ and $x = 2$. We reject $x = -12$, since a negative solution does not make sense in this context. If $x = 2$, the new garden will have dimensions 6 feet by 8 feet, which does give the desired area of 48 square feet.

(d) The Hamburglar steals a hamburger from a restaurant and flees in his getaway car driving due north on a straight road going 60 miles per hour. The police arrive, and after doing an initial investigation, they chase the Hamburglar in their squad car going 80 miles per hour. If the police begin their pursuit half an hour after the Hamburglar left and neither vehicle changes speeds, how far along the road do the police catch up with him?

Solution:
Here, we let t be the time that the Hamburglar leaves the restaurant. Using the base equation $d = rt$ to relate distance with rate and time, we get equations representing the distance of the Hamburglar and the police from the restaurant at a given time:
$d = 60t$ for the Hamburglar, and $d = 80(t - \frac{1}{2}) = 80t - 40$ for the police. Notice that the $-\frac{1}{2}$ term is there to account the half an hour difference in their departure times.
Since we are interested in where the police catch up to the Hamburglar, we consider where the distance traveled by each is the same. Here, $60t = 80t - 40$, or $40 = 20t$, so $t = 2$. That is, the police catch up with the Hamburglar exactly two hours after he leaves the restaurant. Both he and the police have traveled $(60)(2) = 120$ miles at this time.

8. Solve for r in the equation $r^2 - 4(qs)^2 = 0$

Solution: $r^2 - 4q^2s^2 = 0$, or $r^2 = 4q^2s^2$, so, taking the square root or both sides:
$r = \pm\sqrt{4q^2s^2} = \pm 2qs$

9. Solve for R in the following equation: $F = \dfrac{\pi P R^4}{8vL}$

   Solution: Multiplying by $8vL$, $8FvL = \pi P R^4$.

   Then dividing, $\frac{8FvL}{\pi P} = R^4$, so $R = \sqrt{\dfrac{8FvL}{\pi P}}$