Math 127
Exam 3 Practice Problems

1. True or False:

   (a) Any two distinct points in the plane determine exactly one line.
   True. This is a fairly familiar fact from Geometry.

   (b) Any line can be written in the form $y = mx + b$.
   False. This is a bit tricky, but vertical lines cannot be put into the from $y = mx + b$.

   (c) The graph of any circle is symmetric with respect to the origin.
   False. Only circles centered at the origin are symmetric with respect to the origin.

   (d) If a graph has two points with the same y-coordinate, then it is not the graph of a function $y = f(x)$.
   False. A repeated y coordinate is not a problem. Repeated x-coordinates are what we are worried about. For example, $f(x) = x^2$ has lots of repeated y coordinates ($f(x) = 5$ has even more).

   (e) Every function $y = f(x)$ has at least one x-intercept.
   False. Many functions do not have an x-intercept. For example, $f(x) = 5$ and $f(x) = x^2 + 1$ do not have any x-intercepts.

2. Given the points $A(2, -2)$ and $B(-1, 4)$:

   (a) Find $d(A, B)$
   $d(A, B) = \sqrt{(2 - (-1))^2 + (-2 - 4)^2} = \sqrt{3^2 + (-6)^2} = \sqrt{9 + 36} = \sqrt{45} = 3\sqrt{5}$.

   (b) Find the midpoint of the line segment containing A and B.
   $M = \left(\dfrac{2-1}{2}, \dfrac{-2+4}{2}\right) = \left(\dfrac{1}{2}, \dfrac{2}{2}\right) = \left(\dfrac{1}{2}, 1\right)$

   (c) Find the equation for the line containing A and B in general form.
   $m = \frac{4-(-2)}{-1-2} = \frac{6}{-3} = -2$, so, using the point/slope equation:
   $y + 2 = -2(x - 2) = -2x + 4$
   Thus the line has equation $y = -2x + 2$.

   (d) Find the perpendicular bisector of the line segment containing A and B.
   The perpendicular bisector to a line segment is the line through the midpoint of the segment that meets the segment in a right angle.
   Therefore, it is the line through $M = \left(\dfrac{1}{2}, 1\right)$ with slope $\frac{1}{2}$ (the negative reciprocal of $m = -2$)
   By point slope, this is: $y - 1 = \frac{1}{2}(x - \frac{1}{2}) = \frac{1}{2}x - \frac{1}{4}$.
   Hence, the line has equation $y = \frac{1}{2}x + \frac{3}{4}$.

   (e) Find the equation for the circle centered at B containing the point A.
   From part (a) above, $r = 3\sqrt{5}$ and $C = (-1, 4)$.
   Therefore, the circle has equation $(x + 1)^2 + (y - 4)^2 = 45$

   (f) Find an equation for the vertical line containing B.
   $x = -1$

   (g) Find an equation for the horizontal line containing A.
   $y = -2$

3. Find the equation for the following circles:

   (a) The circle with center $(4, -5)$ and radius $\sqrt{15}$
   The circle has equation $(x - 4)^2 + (y + 5)^2 = 15$

   (b) The circle with diameter passing through the points $(2, -2)$ and $(-4, -2)$
   Notice that the distance between these point is: $d(A, B) = \sqrt{(2 - (-4))^2 + (-2 - (-2))^2} = \sqrt{6^2 + 0^2} = \sqrt{36} = 6$.
   Thus the radius is *half* this distance, or $r = 3$ and the center of the circle if the midpoint of the line segment between these points, $C = \left(\frac{2+(-4)}{2}, \frac{-2+-2}{2}\right) = (-1, -2)$.
   Therefore, the circle has equation $(x + 1)^2 + (y + 2)^2 = 9$

(c) The circle with center $(2, 1)$ and passing through the point $(5, 5)$

Notice that the distance between these point is: $d(A, B) = \sqrt{(5-2)^2 + (5-1)^2} = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$.

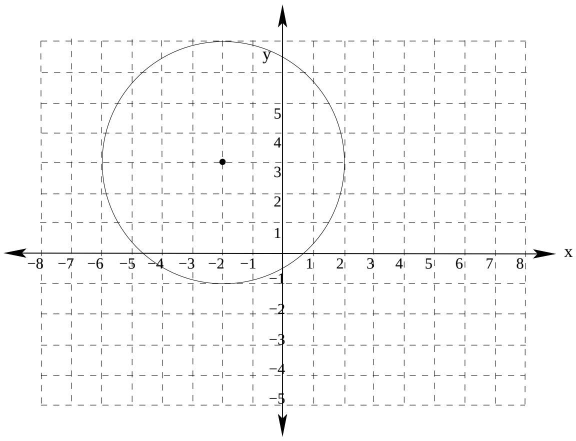
Therefore, $r = 5$ and $C = (2, 1)$, so the circle has equation $(x-2)^2 + (y-1)^2 = 25$

4. Graph the circle with equation $x^2 + y^2 + 4x - 6y - 3 = 0$

Rearranging the terms and completing the square: $x^2 + 4x + \quad + y^2 - 6y + \quad = 3$

Therefore, $x^2 + 4x + 4 + y^2 - 6y + 9 = 3 + 4 + 9$, or $(x+2)^2 + (y-3)^2 = 16$

Thus this circle has center $(-2, 3)$ and radius $r = 4$, so the graph of the circle is:



5. Determine whether or not the following equations are symmetric with respect to the x-axis, y-axis, or the origin.

(a) $y = x^4 - x^2$

Substituting $-x$ for x: $y = (-x)^4 - (-x)^2 = x^4 - x^2$. Thus the graph of this equation is symmetric with respect to the y-axis.

Substituting $-y$ for y: $-y = x^4 - x^2$ or $y = -x^4 + x^2$, so the graph of this equation is *not* symmetric with respect to the x-axis.

Finally, substituting $-x$ for x *and* $-y$ for y: $-y = (-x)^4 - (-x)^2$, or $y = -x^4 + x^2$ so the graph of this equation is *not* symmetric with respect to the origin.

(b) $y = x^3 - 2x$

Substituting $-x$ for x: $y = (-x)^3 - 2(-x) = -x^3 + 2x$, so the graph of this equation is *not* symmetric with respect to the y-axis.

Substituting $-y$ for y: $-y = x^3 - 2x$, or $y = -x^3 + 2x$, so the graph of this equation is *not* symmetric with respect to the y-axis.

However, substituting $-x$ for x *and* $-y$ for y: $-y = (-x)^2 - 2(-x) = -x^3 + 2x$, or, dividing both sides by -1, $y = x^3 - 2x$ so the graph of this equation is symmetric with respect to the origin.

(c) $x^2 - y^2 = 1$

Here, substituting either $-x$ for x or $-y$ for y or substituting both $-x$ for x *and* $-y$ for y gives back $x^2 - y^2 = 1$ when we simplify, so this equation is symmetric with respect to the y-axis, the x-axis, and with respect to the origin.

(d) $y = 3x - 2$

Substituting $-x$ for x gives: $y = -3x - 2$

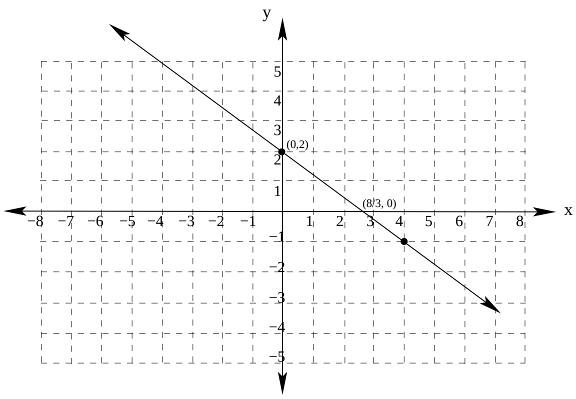
Substituting $-y$ for y gives: $-y = 3x - 2$ or $y = -3x + 2$

Substituting $-x$ for x *and* $-y$ for y gives: $-y = -3x - 2$, or $y = 3x + 2$.

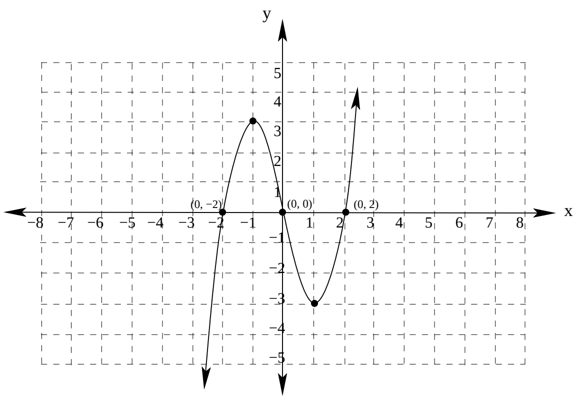
Therefore, this equation is *neither* symmetric with respect to the y-axis, nor the x-axis nor with respect to the origin.

6. Sketch the graphs of the following functions. Be sure to find and label all x and y intercepts.

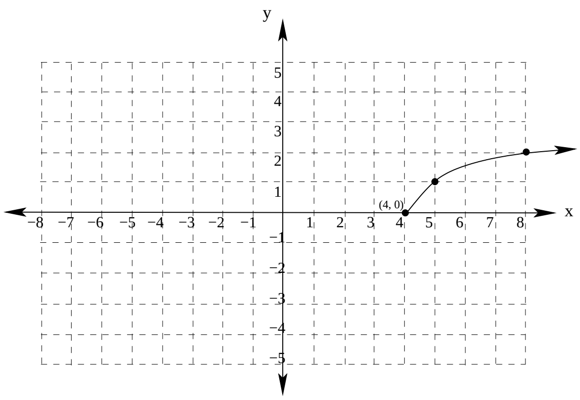
(a) $f(x) = -\frac{3}{4}x + 2$



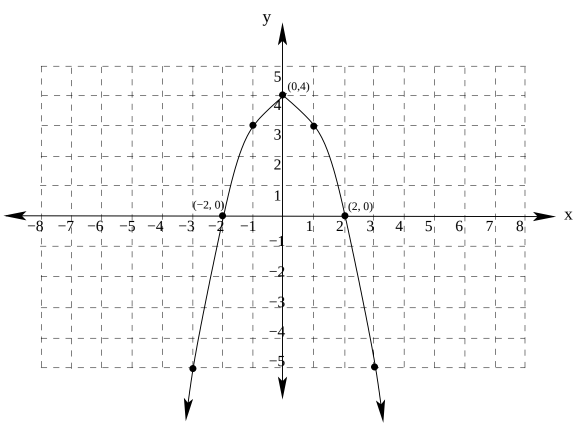
(b) $g(x) = x^3 - 4x$



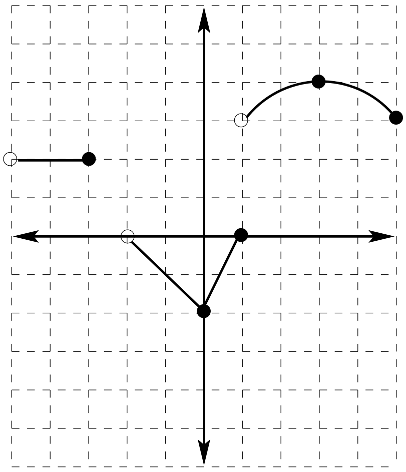
(c) $y = \sqrt{x - 4}$



(d) $y = 4 - x^2$



7. For the given graph of $f(x)$, find the following:



(a) $f(0) = -2$

(b) $f(3) = 4$

(c) x, when $f(x) = 2$

$f(x) = 2$ for every x in the interval $(-5, 3]$

(d) The domain of f

The domain of f is $(-5, -3] \cup (-2, 5]$

(e) The range of f

The range of f is $[-2, 0] \cup \{2\} \cup [3, 4]$

(f) The intervals where f is decreasing.

f is decreasing on $(-2, 0] \cup [3, 5]$

8. Let $f(x) = x^2 - 2x$. Find and simplify the following:

(a) $f(2)$, and $f(\frac{2}{3})$
$f(2) = 2^2 - 2(2) = 4 - 4 = 0$ and $f(\frac{2}{3}) = (\frac{2}{3})^2 - 2(\frac{2}{3}) = \frac{4}{9} - \frac{4}{3} = \frac{4}{9} - \frac{12}{9} = -\frac{8}{9}$

(b) $f(a+3)$
$f(a+3) = (a+3)^2 - 2(a+3) = a^2 + 6a + 9 - 2a - 6 = a^2 + 4a + 3 = (a+1)(a+3)$

(c) $f(2a-1)$
$f(2a-1) = (2a-1)^2 - 2(2a-1) = 4a^2 - 4a + 1 - 4a + 2 = 4a^2 - 8a + 3 = (2a-1)(2a-3)$

(d) $\dfrac{f(a+h) - f(a)}{h}$

$$\frac{f(a+h) - f(a)}{h} = \frac{(a+h)^2 - 2(a+h) - (a^2 - 2a)}{h} = \frac{a^2 + 2ah + h^2 - 2a - 2h - a^2 + 2a}{h}$$
$$= \frac{2ah + h^2 - 2h}{h} = \frac{h(2a + h - 2)}{h} = 2a + h - 2$$

9. Determine whether or not the following are functions:

(a) $\{(3,4), (5,7), (2,-1), (6,8), (8,6)\}$
Since there are no repeated x-coordinates in this set or ordered pairs, this is a function.

(b) $\{(1,2), (3,7), (4,-12), (5,8), (7,2)\}$
Since there are no repeated x-coordinates in this set or ordered pairs, this is a function.

(c) $\{(1,2), (2,3), (3,4), (4,5), (3,5)\}$
Since there are two ordered pairs with 3 as their x-coordinate, this is **not** a function.

10. Find the domain of the following functions (put your answers in interval notation):

(a) $f(x) = \frac{2x+7}{2x^2-3x-2}$
We just need to make sure that the denominator is not zero.
Notice that $2x^2 - 3x - 2 = (2x+1)(x-2)$, so we need to avoid $2x + 1 = 0$ or $2x = -1$ and $x - 2 = 0$
So $x \neq -\frac{1}{2}$ and $x \neq 2$
Therefore, in interval notation, the domain of f is: $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, 2) \cup (2, \infty)$.

(b) $f(x) = \frac{x^2+x-2}{x^2-4}$
We again need to avoid making the denominator zero, so we can't have $x^2 - 4 = 0$ or $x^2 = 4$.
Therefore, $x \neq \pm 2$.
Therefore, in interval notation, the domain of f is: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

(c) $f(x) = \frac{\sqrt{4-2x}}{x^2-1}$
There are two things to worry about. First, we need the denominator to be non-zero. That is, we can't have $x^2 - 1 = 0$ or $x^2 = 1$.
Therefore, $x \neq \pm 1$.
Next, we can't take the square root of a negative number, so we need $4 - 2x \geq 0$.
That is, $4 \geq 2x$, or $2 \geq x$. Combining these, the domain of f is:
$(-\infty, -1) \cup (-1, 1) \cup (1, 2]$

(d) $f(x) = \frac{4}{\sqrt{3x-5}}$
Here, we need $3x - 5 > 0$, or $3x > 5$. Thus $x > \frac{5}{3}$.
Therefore, the domain is: $(\frac{5}{3}, \infty)$

(e) $f(x) = \frac{\sqrt{3-2x}}{2x^2+x-15}$
There are two things to worry about. First, we need the denominator to be non-zero. That is, we can't have $2x^2 + x - 15 = 0$ or $(2x - 5)(x + 3) = 0$.
Therefore, $x \neq \frac{5}{2}$ or $x \neq -3$.
Next, we can't take the square root of a negative number, so we need $3 - 2x \geq 0$.
That is, $3 \geq 2x$, or $\frac{3}{2} \geq x$. Combining these, the domain of f is:
$(-\infty, -3) \cup (-3, \frac{3}{2}]$

11. Given that $f(x) = \sqrt{2x-2}$ and $g(x) = \dfrac{4}{3x-2}$

(a) Find $g(6)$ and $f(3a + 1)$
$g(6) = \frac{4}{3(6)-2} = \frac{4}{16} = \frac{1}{4}$.
$f(3a + 1) = \sqrt{2(3a + 1) - 2} = \sqrt{6a + 2 - 2} = \sqrt{6a}$

(b) Find $\dfrac{g}{f}(3)$
$f(3) = \sqrt{2(3) - 2} = \sqrt{4} = 2$
$g(3) = \frac{4}{3(3)-2} = \frac{4}{7}$
$\dfrac{g}{f}(3) = \dfrac{g(3)}{f(3)} = \dfrac{\frac{4}{7}}{2} = \dfrac{4}{7} \cdot \dfrac{1}{2} = \dfrac{2}{7}$

(c) Find $f \circ g(2)$
$g(2) = \frac{4}{3(2)-2} = \frac{4}{4} = 1$
$f \circ g(2) = f(g(2)) = f(1) = \sqrt{2(1) - 2} = \sqrt{0} = 0$

12. Given that $f(x) = \sqrt{3x-2}$ and $g(x) = x^2 - 4$

(a) Find $g \circ f(x)$
$g \circ f(x) = g(f(x)) = (\sqrt{3x-2})^2 - 4 = 3x - 2 - 4 = 3x - 6 = 3(x - 2)$

(b) Find $f \circ g(x)$
$f \circ g(x) = f(g(x)) = \sqrt{3(x^2 - 4) - 2} = \sqrt{3x^2 - 12 - 2} = \sqrt{3x^2 - 14}$

(c) Find the domain of $g \circ f(x)$. Give your answer in interval notation.
To find the domain of $g \circ f(x) = g(f(x))$, we first find the domain of f:
$3x - 2 \geq 0$, so $3x \geq 2$ or $x \geq \frac{2}{3}$.
Next, notice that g is never undefined.
Therefore, the domain of $g \circ f(x)$ is $[\frac{2}{3}, \infty)$

(d) Find the domain of $\dfrac{f}{g}$. Give your answer in interval notation.
To be in the domain of $\dfrac{f}{g}$, we need $f(x)$ to be defined, and $g(x)$ to be defined and non-zero.
Therefore, we need $3x - 2 \geq 0$, or $3x \geq 2$, hence $x \neq \frac{2}{3}$.
We also need $x^2 - 4 \neq 0$, or $x \neq \pm 2$
Hence the domain of $\dfrac{f}{g}$ is $[\frac{2}{3}, 2) \cup (2, \infty)$

(e) Find $\dfrac{g(a+h)-g(a)}{h}$. Simplify your answer.

$$\frac{g(a+h)-g(a)}{h} = \frac{(a+h)^2-4-(a^2-4)}{h} = \frac{a^2+2ah+h^2-4-a^2+4}{h} = \frac{2ah+h^2}{h} = \frac{h(2a+h)}{h} = 2a+h$$

13. An oil well off the Gulf Coast is leaking, with the leak spreading oil over the surface in the shape of a circle. At any time t, in minutes, after the beginning of the leak, the radius of the circular oil slick on the surface is $r(t) = 4t$ feet. Let $A(r) = \pi r^2$ represent the area of the circle of radius r.

(a) Find $(A \circ r)(t)$

Since $r(t) = 4t$ and $A(r) = \pi r^2$, $(A \circ r)(t) = \pi(4t)^2 = 16\pi t^2$

(b) Explain what $(A \circ r)(t)$ is in practical terms.

$(A \circ r)(t)$ gives the area of the oil as a function of time in minutes.