Math 127
Exam 4 Practice Problem Solutions

1. Solve the following systems of linear equations. Then graph the equations involved to show that your answer is reasonable.

(a) $\begin{cases} y = 3x - 2 \\ x + 2y = 5 \end{cases}$

Since one equation is already solved in terms of y, we will use the substitution method:
$x + 2(3x - 2) = 5$, or $x + 6x - 4 = 5$. Therefore, $7x = 9$, or $x = \frac{9}{7}$.

Resubstituting this value, we have: $y = 3(\frac{9}{7}) - 2 = \frac{27}{7} - \frac{14}{7} = \frac{13}{7}$.

Therefore, the solution consists of the point: $(\frac{9}{7}, \frac{13}{7})$ [I'll let you make the related graphs yourselves...]

(b) $\begin{cases} 3x - 2y = 7 \\ x + 2y = 5 \end{cases}$

For this equation, we will use elimination, since two terms are already opposites. Adding these equations gives: $4x = 12$, or $x = 3$. So if we substitute this value back into the second equation, we have:

$3 + 2y = 5$, or $2y = 2$. Thus $y = 1$.

Therefore, the solution consists of the point: $(3, 1)$

(c) $\begin{cases} x - 3y = 2 \\ 6y - 2x = 5 \end{cases}$

We will again use the elimination method. If we multiply the first equation by 2 and then rearrange so that the terms match up, we have:
$\begin{cases} 2x - 6y = 4 \\ -2x + 6y = 5 \end{cases}$

Adding these gives us: $0 + 0 = 9$ or $0 = 9$ which is always false.

Therefore, we conclude that there is no solution and that the lines represented by these equations are parallel.

(d) $\begin{cases} 5x - 4y = 10 \\ 3x + 5y = 12 \end{cases}$

We will again use the elimination method. If we multiply the first equation by 3 and the second by -5, which gives us:
$\begin{cases} 15x - 12y = 30 \\ -15x - 25y = 60 \end{cases}$

Adding these gives us: $-37y = -30$, or $y = \frac{30}{37}$.
If we substitute this value back into the original first equation, we have:

$3x + 5\left(\frac{30}{37}\right) = 12$, or $3x = 12 - \frac{150}{37} = \frac{444}{37} - \frac{150}{37} = \frac{294}{37}$.
Thus $x = \frac{1}{3}\frac{294}{37} = \frac{98}{37}$.

Therefore, the solution consists of the point: $(\frac{98}{37}, \frac{30}{37})$

(e) $\begin{cases} x - 2y = 5 \\ -2x + 4y = 10 \end{cases}$

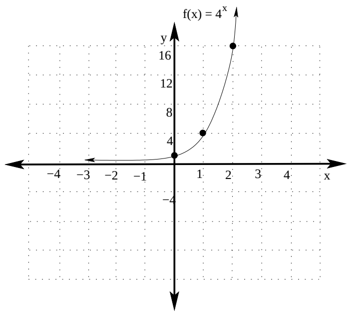
We will again use the elimination method. If we multiply the first equation by 2, then we have:
$\begin{cases} 2x - 4y = 10 \\ -2x + 4y = 10 \end{cases}$

Adding these gives us: $0 + 0 = 20$ or $0 = 20$ which is always false.

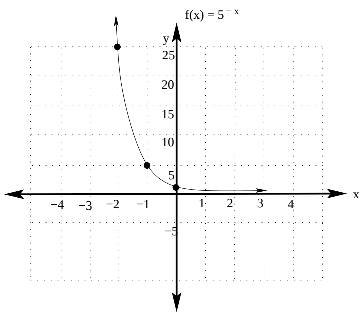
Therefore, we conclude that there is no solution and that the lines represented by these equations are parallel.

2. Graph each of the following functions:

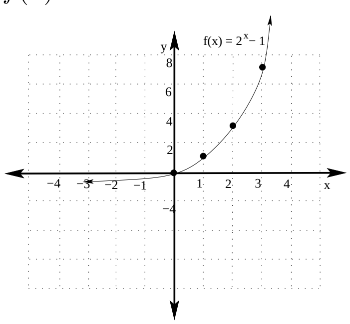
(a) $f(x) = 4^x$



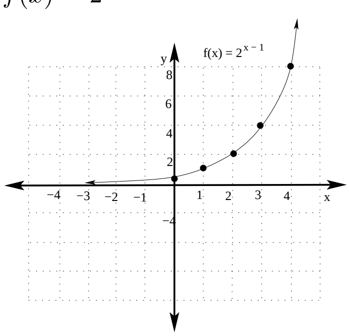
(b) $f(x) = 5^{-x}$



(c) $f(x) = 2^x - 1$



(d) $f(x) = 2^{x-1}$



3. Solve the following equations:

(a) $3^{2-3x} = 3^{2x+1}$

Using the fact that exponential functions are one-to-one, we must have:
$2 - 3x = 2x + 1$, or $5x = 1$.
Thus $x = \frac{1}{5}$.

(b) $5^{4x} = 5^{3x-12}$

Using the fact that exponential functions are one-to-one, we must have:
$4x = 3x - 12$, or $x = -12$.

(c) $2^{5x+1} = 4^{3-2x}$

We first rewrite so that both exponentials have the same base: $2^{5x+1} = (2^2)^{3-2x}$ or $2^{5x+1} = 2^{6-4x}$
Using the fact that exponential functions are one-to-one, we must have:
$5x + 1 = 6 - 4x$, or $9x = 5$.
Thus $x = \frac{5}{9}$.

4. Suppose you have \$2,000 to invest.

(a) Find the amount you would have after 5 years if you deposit your \$2,000 in an account that pays 6% annual interest compounded monthly.

Using the compound interest equation $A = P\left(1 + \frac{r}{n}\right)^{nt}$, we have $A = 2,000\left(1 + \frac{.06}{12}\right)^{(5)(12)} = 2,000(1.005)^{60} \approx$ \$2,697.70

(b) Find the amount you would have after 5 years if you deposit your \$2,000 in an account that pays 4% annual interest compounded quarterly.

Using the compound interest equation $A = P\left(1 + \frac{r}{n}\right)^{nt}$, we have $A = 2,000\left(1 + \frac{.04}{4}\right)^{(5)(4)} = 2,000(1.01)^{20} \approx$ \$2,440.38

(c) Find the amount you would have after 5 years if you deposit your \$2,000 in an account that pays 5% annual interest compounded continuously.

Using the continuous compounding equation $A = Pe^{rt}$, we have $A = 2,000e^{(.05)(5)} = 2,000e^{.25} \approx$ \$2,568.05

5. Translate each of the following expressions into exponential form:

(a) $\log_5 x = y$
$5^y = x$

(b) $\log_z 5 = y$
$z^y = 5$

(c) $\log_y x = 5$
$y^5 = x$

6. Find the *exact* value of each of the following:

(a) $\log_{21}(1) = 0$ since $21^0 = 1$

(b) $\log_7(0)$ is undefined since there is no exponent x such that $7^x = 0$.

(c) $\log_2(\frac{1}{8}) = -3$, since $2^{-3} = \frac{1}{8}$.

(d) $\log_3(27) = 3$, since $3^3 = 27$

(e) $\ln(e^2) = 2$ since ln represents $\log_e$.

(f) $\log .0001 = -4$ since $10^{-4} = .0001$

(g) $\log_9(27) = 1.5$, since $27 = 3^3 = \left(9^{\frac{1}{2}}\right)^3 = 9^{\frac{3}{2}}$.

(h) $7^{\log_7(\pi)} = \pi$, by the inverse property of logarithms and exponentials.

7. Determine whether the following are True or False:

(a) $\ln\left(\dfrac{x^3}{(x+1)(x-1)}\right) = 3\ln x - \ln(x+1) + \ln(x-1)$

False. Notice that $\ln\left(\dfrac{x^3}{(x+1)(x-1)}\right) = 3\ln x - \ln(x+1) - \ln(x-1)$.

(b) $e^{\ln(x^2+1)} = x^2 + 1$

True. This is the inverse function property of exponential and log functions.

(c) $e^{x^2} \cdot e^{3x} = e^{3x^3}$

False. In fact, $e^{x^2} \cdot e^{3x} = e^{x^2+3x}$. The exponents **add** rather than multiply here.

(d) $\dfrac{\ln(4x)}{\ln(2x)} = \ln 2$

False. This is not a legal simplification. For example, if $x = 1$, then $\dfrac{\ln(4x)}{\ln(2x)} = \dfrac{\ln(4)}{\ln(2)} = 2$, while $\ln 2 \approx .6931$.

(e) $\ln\left(e^{x^2} - 4\right) = x^2 - \ln 4$

False. The inverse function property does not apply here because of the subtraction operation. In fact, if $x = 2$, $\ln\left(e^{2^2} - 4\right) \approx 3.924$, while $2^2 - \ln 4 \approx 2.614$

8. Use properties of logarithms to expand the following expression:

$\log\left(\dfrac{x^4 z^2}{\sqrt[3]{y}}\right)$

$= \log(x^4 z^2) - \log(\sqrt[3]{y}) = \log(x^4) + \log(z^2) - \log(y^{\frac{1}{3}})$

$= 4\log x + 2\log z - \frac{1}{3}\log y$

9. Use the laws of logarithms to expand the expression: $\ln\left(\dfrac{x^2(x-1)^{\frac{5}{2}}}{(x-4)^3}\right)$

$$\ln\left(\frac{x^2(x-1)^{\frac{5}{2}}}{(x-4)^3}\right) = \ln x^2 + \ln(x-1)^{\frac{5}{2}} - \ln(x-4)^3 = 2\ln x + \frac{5}{2}\ln(x-1) - 3\ln(x-4)$$

10. Use the properties of logarithms to write the following as a single logarithm:

$\dfrac{3}{2}\log_b x^3 y^4 - \dfrac{2}{3}\log_b x^4 y^3 - 2\log_b xy$

$= \log_b\left(x^3 y^4\right)^{\frac{3}{2}} - \log_b\left(x^4 y^3\right)^{\frac{2}{3}} - \log_b(xy)^2$

$= \log_b\left(x^{\frac{9}{2}} y^6\right) - \left[\log_b\left(x^{\frac{8}{3}} y^2\right) + \log_b(x^2 y^2)\right]$

$= \log_b\left(x^{\frac{9}{2}} y^6\right) - \log_b\left(x^{\frac{8}{3}} y^2 x^2 y^2\right)$

$= \log_b\left(x^{\frac{9}{2}} y^6\right) - \log_b\left(x^{\frac{14}{3}} y^4\right)$

$= \log_b \dfrac{x^{\frac{9}{2}} y^6}{x^{\frac{14}{3}} y^4} = \log_b x^{-\frac{1}{6}} y^2$

11. Use the change of base formula to approximate the following:

(a) $\log_5 10$

By the change of base formula: $\log_b u = \dfrac{\log_a u}{\log_a b}$

Thus $\log_5 10 = \dfrac{\ln 10}{\ln 5} \approx 1.4307$

(b) $\log_9 12$

By the change of base formula: $\log_b u = \dfrac{\log_a u}{\log_a b}$

Thus $\log_9 12 = \dfrac{\ln 12}{\ln 9} \approx 1.1309$

(c) $\log_{15} 7$

By the change of base formula: $\log_b u = \dfrac{\log_a u}{\log_a b}$

Thus $\log_{15} 7 = \dfrac{\ln 7}{\ln 15} \approx 0.7186$

12. Convert the following functions to base e exponentials:

(a) $f(x) = 100 \cdot 4^x$

$f(x) = 100 \cdot 4^x = 100e^{(\ln 4)x} \approx 100e^{1.3863x}$

(b) $g(x) = 50 \cdot \left(\frac{1}{4}\right)^x$

$$g(x) = 50 \cdot \left(\tfrac{1}{4}\right)^x = 50e^{\ln\left(\frac{1}{4}\right)x} \approx 50e^{-1.3863x}$$

(c) $h(x) = 450 \cdot 5^{-2x}$

$$h(x) = 450 \cdot 5^{-2x} = 450 \cdot \left(\tfrac{1}{25}\right)^x = 450e^{\ln\left(\frac{1}{25}\right)x} = 450e^{-3.2189x}$$

13. Solve the following equations (give exact answers whenever possible):

(a) $e^{3x-2} = e^{4-5x}$
By the one-to-one property of exponentials, $3x - 2 = 4 - 5x$.
Then $8x = 6$, so $x = \frac{6}{8}$ or $x = \frac{3}{4}$

(b) $9^{2x} = 27(3)^{2x+1}$
Notice that all the terms can be written as powers of 3:
$\left(3^2\right)^{2x} = 3^3(3)^{2x+1}$, or $3^{4x} = 3^{2x+4}$.
Therefore, $4x = 2x + 4$, so $2x = 4$, or $x = 2$.

(c) $\log_2(3x^2 - 3) = \log_2(x^2 + x)$
Since the base on both logarithms are the same, by the one-to-one property, $3x^2 - 3 = x^2 + x$.
Moving everything to one side, we have: $2x^2 - x - 3 = 0$, or $(2x - 3)(x + 1) = 0$
Therefore, either $2x = 3$ or $x = -1$.
That is, either $x = \frac{3}{2}$, or $x = -1$.
However, notice that $x = -1$ does not check since you cannot take the logarithm of zero.
Thus the only solution is $x = \frac{3}{2}$.

(d) $\log_5(x^2 + 21) = 2$
Rewriting in exponential form, we have $5^2 = x^2 + 21$, or $25 = x^2 + 21$.
Therefore, $x^2 - 4 = 0$, or $(x + 2)(x - 2) = 0$. Thus either $x = 2$, or $x = -2$.
Notice that both of these solutions check, since $x^2 + 21$ is positive when $x = \pm 2$.

(e) $\log_2(2x) + \log_2(x - 3) = 3$
Using the properties of logarithms to write this as a single logarithm, we have:
$\log_2((2x)(x - 3)) = 3$, or $\log_2(2x^2 - 6x) = 3$.
Changing this to exponential form, we have $2^3 = 2x^2 - 6x$.
Moving everything to one side gives: $2x^2 - 6x - 8 = 0$, or $(2x - 8)(x + 1) = 0$. Thus either $2x = 8$, so $x = 4$, or $x = -1$.
However, notice that $x = -1$ does not check since you cannot take the logarithm of a negative quantity.
Thus the only solution is $x = 4$.

(f) $\log(\sqrt[4]{x + 1}) = \dfrac{1}{2}$
Re-writing in exponential form gives us: $10^{\frac{1}{2}} = \sqrt[4]{x + 1}$
Then $\left(10^{\frac{1}{2}}\right)^4 = \left(\sqrt[4]{x + 1}\right)^4$, or $10^2 = x + 1$

Therefore $100 = x + 1$ or $x = 99$. (Notice this this solution does check since $\sqrt[4]{99 + 1} = \sqrt[4]{100} = 100^{\frac{1}{2}=10^{\frac{1}{2}}}$, and $\log 10^{\frac{1}{2}} = \frac{1}{2}$).

(g) $e^{2x-1} = 3$
Taking the natural log of both sides: $\ln\left(e^{2x-1}\right) = \ln 3$, or $2x - 1 = \ln 3$.
Therefore, $2x = \ln 3 + 1$, so $x = \frac{\ln 3 + 1}{2}$

(h) $4^{2x-1} = 3^{5x}$
Since the bases are different , we must solve this by taking the logarithm of both sides:
$\ln\left(4^{2x-1}\right) = \ln\left(3^{5x}\right)$.
Therefore, $(2x - 1)\ln 4 = 5x \ln 3$, or $(2\ln 4)x - \ln 4 = (5\ln 3)x$.
Getting every term containing an x to one side and the constants on the other side:
$(2\ln 4)x - (5\ln 3)x = \ln 4$, or $x(2\ln 4 - 5\ln 3) = \ln 4$
Hence $x = \frac{\ln 4}{2\ln 4 - 5\ln 3} \approx -.5096$.

14. (a) Suppose you invest $10,000 in a savings account that pays 3% annual interest compounded monthly. How much money will be in the account after 6 years?

$A = P(1 + \frac{r}{n})^{nt} = 10,000(1 + \frac{.03}{12})^{(12)(6)} = 10,000(1.0025)^{72} \approx \$11,969.48$

(b) How long would it take $5,000 invested at 6% annual interest compounded continuously to triple?

The continuous interest formula is: $A = Pe^{rt}$. So we have $15,000 = 5,000e^{.06t}$, or $3 = e^{.06t}$, which makes sense since we want our initial investment to triple.
Taking the natural logarithm of both sides gives: $\ln 3 = \ln(e^{.06t}) = .06t$, so $t = \frac{\ln 3}{.06} \approx 18.31$ years.

(c) Find the interest rate needed for an investment of $2,000 to double in 6 years if the interest is compounded quarterly.

Using the compound interest formula $A = P(1 + \frac{r}{n})^{nt}$, we have $4,000 = 2,000(1 + \frac{r}{4})^{(4)(6)}$, or $2 = (1 + \frac{r}{4})^{24}$. Taking the natural log of both sides, $\ln 2 = \ln(1 + \frac{r}{4})^{24} = 24\ln(1 + \frac{r}{4})$, so $\frac{\ln 2}{24} = \ln(1 + \frac{r}{4})$. Exponentiating both sides, we than have $e^{\frac{\ln 2}{24}} = e^{\ln(1+\frac{r}{4})} = 1 + \frac{r}{4}$.

Hence $e^{\frac{\ln 2}{24}} - 1 = \frac{r}{4}$, therefore $4(e^{\frac{\ln 2}{24}} - 1) = r$, so $r \approx .1172$, or $\%11.72$

15. Suppose that a culture of bacteria that initially has 500 cells grows to 10,000 cells in 12 hours.

(a) Find a function $f(t)$ that gives the number of cells in the culture as a function of time (in hours), assuming that this population grows continuously and exponentially.

Using the continuous exponential growth equation, $A = Pe^{rt}$, we see that $A = 10,000$, $p = 500$, and $t = 12$, or $10,000 = 500e^{12r}$. Therefore, $20 = e^{12r}$, so $\ln(20) = 12r$, and hence $r \approx .2496$.
Therefore, our function modeling the growth of this bacterial culture is: $f(t) = 500e^{.2496t}$.

(b) How long will it take for the culture to reach 1,000,000 cells?
Using the function $f(t) = 500e^{.2496t}$ found above, we solve $1,000,000 = 500e^{.2496t}$ for t.
Then $2000 = e^{.2496t}$, so $\ln(2000) = .2496t$, so $t = \frac{\ln(2000)}{.2496} \approx 30.45$ hours

16. Suppose a certain substance has a half life of 47 years. If you start with 100 grams of the substance, how long will it take for the amount to be reduced to 50 grams? How long will it take for the amount to be reduced to 12 grams?

First notice that since we start with 100 grams of the substance and the half-life is 47 years, it will take 47 years for the initial 100 grams to be reduced down to 50 grams.

Next, to find the time needed for the initial 100 grams to be reduced down to 12 grams, we will need to construct a model. Let $f(t) = A_0e^{kt}$ be our exponential decay model with t in years. Then we have:
$\frac{1}{2}A_0 = A_0e^{k \cdot 47}$, or $\frac{1}{2} = e^{47k}$.
Then $\ln\left(\frac{1}{2}\right) = \ln\left(e^{47k}\right)$, or $\ln\left(\frac{1}{2}\right) = 47k$.

Thus $k = \frac{\ln\left(\frac{1}{2}\right)}{47} \approx -0.01475$.

From this, $f(t) = A_0e^{-0.01475t}$, or, starting with 100 grams and ending with 12 grams:
$12 = 100e^{-0.01475t}$, or $\frac{12}{100} = e^{-0.01475t}$. Thus $\ln\left(\frac{12}{100}\right) = -0.01475t$.

Hence $t = \frac{\ln\left(\frac{12}{100}\right)}{-0.01475} \approx 143.75$.
Therefore, it would take approximately 143.75 years for the initial 100 grams to be reduced down to 12 grams.