

Long Division Examples:

1. Find: $\frac{5x^3 - 4x^2 + 7x - 13}{x + 2}$

$$\begin{array}{r}
 5x^2 - 14x + 35 \\
 x + 2 \overline{) 5x^3 - 4x^2 + 7x - 13} \\
 \underline{-5x^3 - 10x^2} \\
 -14x^2 + 7x \\
 \underline{14x^2 + 28x} \\
 35x - 13 \\
 \underline{-35x - 70} \\
 -83
 \end{array}$$

$$= 5x^3 - 4x^2 + 7x - 13 + \frac{-83}{x+2}$$

2. Find: $\frac{2x^4 - x^2 + 7x - 3}{x - 3}$

$$\begin{array}{r}
 2x^3 + 6x^2 + 17x + 58 \\
 x - 3 \overline{) 2x^4 + 7x - 3} \\
 \underline{-2x^4 + 6x^3} \\
 6x^3 - x^2 \\
 \underline{-6x^3 + 18x^2} \\
 17x^2 + 7x \\
 \underline{-17x^2 + 51x} \\
 58x - 3 \\
 \underline{-58x + 174} \\
 171
 \end{array}$$

$$= 2x^3 + 6x^2 + 17x + 58 + \frac{171}{x-3}$$

3. Find: $\frac{4x^5 - 2x^4 + x^2 - 11}{2x + 3}$

$$\begin{array}{r}
 2x^4 - 4x^3 + 6x^2 - \frac{17}{2}x + \frac{51}{4} \\
 2x + 3 \overline{) 4x^5 - 2x^4 - 11} \\
 \underline{-4x^5 - 6x^4} \\
 -8x^4 \\
 \underline{8x^4 + 12x^3} \\
 12x^3 + x^2 \\
 \underline{-12x^3 - 18x^2} \\
 -17x^2 \\
 \underline{17x^2 + \frac{51}{2}x} \\
 \frac{51}{2}x - 11 \\
 \underline{-\frac{51}{2}x - \frac{153}{4}} \\
 -\frac{197}{4}
 \end{array}$$

$$= 2x^4 - 4x^3 - 6x^2 - \frac{17}{2}x + \frac{51}{4} + \frac{-197}{4(2x+3)}$$

4. Find: $\frac{3x^4 - 5x^3 + 7x^2 - 5}{x^2 + 2x - 1}$

$$\begin{array}{r}
 3x^2 - 11x + 32 \\
 x^2 + 2x - 1 \overline{) 3x^4 - 5x^3 + 7x^2 - 5} \\
 \underline{-3x^4 - 6x^3 + 3x^2} \\
 -11x^3 + 10x^2 \\
 \underline{11x^3 + 22x^2 - 11x} \\
 32x^2 - 11x - 5 \\
 \underline{-32x^2 - 64x + 32} \\
 -75x + 27
 \end{array}$$

$$= 3x^2 - 11x + 32 + \frac{-75x+27}{x^2+2x-1}$$

Synthetic Division Examples:

1. Find: $\frac{4x^2 + 3x - 5}{x - 2}$

$$\begin{array}{r|rrr}
 & 4 & 3 & -5 \\
 2 & & 8 & 22 \\
 \hline
 & 4 & 11 & 17
 \end{array}$$

$$= 4x + 11 + \frac{17}{x-2}$$

2. Find: $\frac{x^3 - 3x^2 + 4x - 1}{x + 3}$

$$\begin{array}{r|rrrr}
 & 1 & -3 & 4 & -1 \\
 -3 & & -3 & 18 & -66 \\
 \hline
 & 1 & -6 & 22 & -67
 \end{array}$$

$$= x^2 - 6x + 22 + \frac{-67}{x+3}$$

3. Evaluate $f(4)$ if $f(x) = x^3 + 7x - 5$

$$\begin{array}{r|rrrr}
 & 1 & 0 & 7 & -5 \\
 4 & & 4 & 16 & 92 \\
 \hline
 & 1 & 4 & 23 & 87
 \end{array}$$

So $f(4) = 87$

4. Evaluate $f(-1)$ if $f(x) = x^4 - 2x^3 + 5x^2 + 7x - 11$

$$\begin{array}{r|rrrrr}
 & 1 & -2 & 5 & 7 & -11 \\
 -1 & & -1 & 3 & -8 & 1 \\
 \hline
 & 1 & -3 & 8 & -1 & -10
 \end{array}$$

So $f(-1) = -10$

The Remainder Theorem: If a polynomial is divided by the factor $x - c$, the the value of the remainder r is equal to $f(c)$.

The Factor Theorem: For any polynomial $f(x)$:

(a) If $f(c) = 0$, then $(x - c)$ is a factor of $f(x)$. That is, $f(x) = (x - c)q(x)$ for some polynomial $q(x)$.

(b) If $(x - c)$ is a factor of $f(x)$, then $f(c) = 0$.

The Rational Root Theorem: If $f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ is a polynomial with integer coefficients and $\frac{p}{q}$ is a rational zero of $f(x)$, then p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n .

Example: Let $f(x) = 3x^3 + 7x^2 + 26x - 7$. Using the rational root theorem, since $a_0 = -7$ and $a_n = 4$, we must have:

$p = \pm 1, \pm 7$ and $q = \pm 1, \pm 2, \pm 4$, hence $\frac{p}{q} = \pm 7, \pm \frac{7}{2}, \pm \frac{7}{4}, \pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}$ is a complete list of all the possible rational zeros of $f(x)$.

We will use synthetic division to show that $x = \frac{1}{4}$ is a zero of this polynomial function.

$$\begin{array}{r|rrrr} \frac{1}{4} & 4 & 7 & 26 & -7 \\ & & 1 & 2 & 7 \\ \hline & 4 & 8 & 28 & 0 \end{array}$$

From this, we see that $f(x) = (x - \frac{1}{4})(4x^2 + 8x^2 + 28) = 4(x - \frac{1}{4})(x^2 + 2x^2 + 7)$

To find the other zeros of this polynomial, we can apply the quadratic formula to the remaining quadratic polynomial $y = x^2 + 2x^2 + 7$.

Then $x = \frac{-2 \pm \sqrt{4 - 4(1)(7)}}{2(1)} = \frac{-2 \pm \sqrt{-24}}{2} = -1 \pm i\sqrt{6}$. Thus $x = -1 + i\sqrt{6}$ and $x = -1 - i\sqrt{6}$ are zeros of this polynomial.

To verify this, suppose we multiply the related factors:

$$(x + 1 - i\sqrt{6})(x + 1 + i\sqrt{6}) = x^2 + x - i\sqrt{6}x + x + 1 - i\sqrt{6} + i\sqrt{6} + i\sqrt{6} + i\sqrt{6} - i^2 \cdot 6 = x^2 + 2x + 1 - 6i^2 = x^2 + 2x + 1 + 6 = x^2 + 2x + 7$$

Hence the zeros of this polynomial are $x = \frac{1}{4}$, $x = -1 + i\sqrt{6}$, and $x = -1 - i\sqrt{6}$.

Notes:

1. If $f(x)$ is a polynomial of degree n , then, counting multiple roots separately, $f(x)$ will have n roots.
2. Complex roots always come in pairs. That is, if $a + bi$ is a root of a polynomial with real coefficients, then the conjugate $a - bi$ is also a root of the polynomial.