Quadratic Functions

Definition: A **quadratic function** is any function whose correspondence can be written in the form $f(x) = ax^2 + bx + c$ with $a \neq 0$.

Graphing Quadratic Functions:

It is important to notice that **every** quadratic function can be thought of as a transformation of the quadratic function $f(x) = x^2$. Therefore, every quadratic function has a graph that is in the shape of a parabola – that is, in the shape of the graph of $y = x^2$, but it may be shifted horizontally and/or vertically, and it may be compressed, stretched, and/or reflected.

To understand which transformations have been applied to a given quadratic function, we need to put the function into *standard form* (also called *vertex form*). That is, in the form: $f(x) = a(x - h)^2 + k$.

Once a quadratic function has been put into standard (vertex) form, we can then "read off" the transformations applied in order to fer $f(x)$ and graph it accordingly.

Example: Suppose $f(x) = 2x^2 + x - 6$. To put $f(x)$ into standard form, we will use a modified version of completing the square.

Step 1: Factor out the leading coefficient a from the terms involving x (here $a = 2$)

$$f(x) = 2\left(x^2 + \frac{1}{2}x\right) - 6$$

Step 2: Add the constant needed to make the quadratic factor as a perfect square, and, to keep the value of the expression the same, **subtract** a times the constant you added from the expression. As before, the constant we add will be half the coefficient of the x term squared.

$$f(x) = 2\left(x^2 + \frac{1}{2}x + \left(\frac{1}{4}\right)^2\right) - 6 - 2 \cdot \left(\frac{1}{4}\right)^2$$

or $f(x) = 2\left(x^2 + \frac{1}{2}x + \frac{1}{16}\right) - 6 - 2 \cdot \frac{1}{16}$

Step 3: Factor the perfect square term and simplify the constant on the outside.

$$f(x) = 2\left(x + \frac{1}{4}\right)^2 - 6 - \frac{1}{8}$$

or $f(x) = 2\left(x + \frac{1}{4}\right)^2 - \frac{49}{8}$

This quadratic function is now in standard form: $f(x) = a(x - h)^2 + k$. Here, $a = 2$, $h = -\frac{1}{4}$ and $k = -\frac{49}{8}$

So what does this tell us about the graph of $f(x)$? First and foremost, (h, k) is the vertex of the graph, so this function is a parabola with vertex $\left(-\frac{1}{4}, -\frac{49}{8}\right)$. Notice that this is due to the fact that $y = x^2$ has vertex $(0, 0)$, and we have shifted the graph h units horizontally and k units vertically.

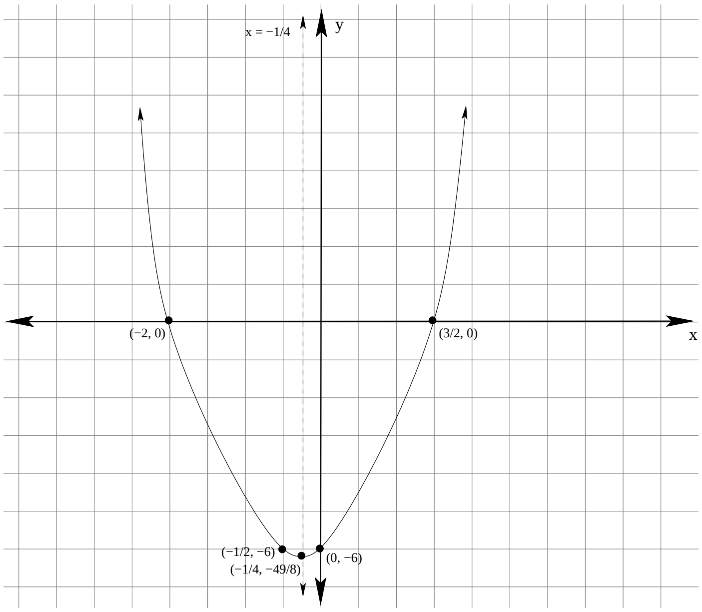
Next, a tells us that the graph has been stretched vertically by a factor or two. Since $a > 0$, there has **not** been a vertical reflection (when $a < 0$ the graph is reflected across the x-axis and will open **downward** rather than *upward*).

It is often useful to find the intercepts of our graph. To find the y-intercept, we find $f(0) = -6$. To find any x-intercepts, we can apply the quadratic formula to the original function after setting $y = 0$: $0 = 2x^2 + x - 6$, so $x = \frac{-1 \pm \sqrt{1^2 - 4(2)(-6)}}{2(2)} = \frac{-1 \pm \sqrt{49}}{4} = -\frac{1}{4} \pm \frac{7}{4}$, so $x = -2$, or $x = \frac{3}{2}$ [in fact, we could have found these by factoring...]

Thus we know that the following point are all on the graph of $f(x)$: $\left(-\frac{1}{4}, -\frac{49}{8}\right)$, $(0, -6)$, $(-2, 0)$, and $\left(\frac{3}{2}, 0\right)$.

Finally, since our original graph $y = x^2$ was symmetric with respect to the y-axis, the graph of $f(x)$ will be symmetric with respect to the vertical line through its vertex. We call this vertical line the **axis of symmetry**. It always has $x = -\frac{b}{2a}$ as its equation. In this example, the axis of symmetry is: $x = -\frac{1}{2(2)} = -\frac{1}{4}$. [Since $(0, -6)$ is on the graph of $f(x)$, what other point do we know is on the graph using symmetry?]

Here is the graph of $f(x)$:



Example 2: Use the procedure described above to graph the function $f(x) = -16x^2 + 144x + 100$