Math 127
Exam 3 Practice Problems

1. True or False:
   (a) Any two distinct points in the plane determine exactly one line.
   True. This is a fairly familiar fact from Geometry.
   (b) Any line can be written in the form $y = mx + b$.
   False. This is a bit tricky, but vertical lines cannot be put into the from $y = mx + b$.
   (c) The graph of any circle is symmetric with respect to the origin.
   False. Only circles centered at the origin are symmetric with respect to the origin.
   (d) If a graph has two points with the same y-coordinate, then it is not the graph of a function $y = f(x)$.
   False. A repeated y coordinate is not a problem. Repeated x-coordinates are what we are worried about. For example, $f(x) = x^2$ has lots of repeated y coordinates ($f(x) = 5$ has even more).
   (e) Every function $y = f(x)$ has at least one x-intercept.
   False. Many functions do not have an x-intercept. For example, $f(x) = 5$ and $f(x) = x^2 + 1$ do not have any x-intercepts.

2. Given the points $A(2, -2)$ and $B(-1, 4)$:
   (a) Find $d(A, B)$
   $d(A,B) = \sqrt{(2-(-1))^2 + (-2-4)^2} = \sqrt{3^2 + (-6)^2} = \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}$.
   (b) Find the midpoint of the line segment containing A and B.
   $M = \left(\frac{2-1}{2}, \frac{-2+4}{2}\right) = \left(\frac{1}{2}, \frac{2}{2}\right) = \left(\frac{1}{2}, 1\right)$
   (c) Find the equation for the line containing A and B in general form.
   $m = \frac{4-(-2)}{-1-2} = \frac{6}{-3} = -2$, so, using the point/slope equation:
   $y + 2 = -2(x-2) = -2x + 4$
   Thus the line has equation $y = -2x + 2$.
   (d) Find the equation for the circle centered at B containing the point A.
   From part (a) above, $r = 3\sqrt{5}$ and $C = (-1, 4)$.
   Therefore, the circle has equation $(x+1)^2 + (y-4)^2 = 45$
   (e) Find an equation for the vertical line containing B.
   $x = -1$
   (f) Find an equation for the horizontal line containing A.
   $y = -2$

3. Find the equation for each line described below. Put your final answer in slope/intercept form.
   (a) The line with slope 4 and y-intercept -7
   $y = 4x - 7$
   (b) The line containing the points $(-4, 1)$ and $(3, -7)$
   First, we find the slope of this line: $m = \dfrac{1-(-7)}{-4-3} = -\dfrac{8}{7}$
   Then, we use the point/slope formula: $y - 1 = -\frac{8}{7}(x+4)$ or $y = -\frac{8}{7}x - \frac{32}{7} + 1$
   Thus $y = -\frac{8}{7}x - \frac{25}{7}$
   (c) The line parallel to the line $3x - 4y = 12$ passing through the point $(1, 3)$
   Putting this line into slope intercept form, we have: $4y = 3x - 12$, or $y = \frac{3}{4}x - 3$
   Then, since we are looking for a parallel line, we need a line with slope $m = \frac{3}{4}$ passing through $(1, 3)$.
   Then, using the point/slope formula: $y - 3 = \frac{3}{4}(x-1)$ or $y = \frac{3}{4}x - \frac{3}{4} + 3$
   Thus $y = \frac{3}{4}x + \frac{9}{4}$
   (d) The line perpendicular to the line $5y - 2x = 3$ and having x-intercept -1.
   Putting this line into slope intercept form, we have: $5y = 2x + 3$, or $y = \frac{2}{5}x + \frac{3}{5}$
   Then, since we are looking for a perpendicular line, we need a line with slope $m = -\frac{5}{2}$ passing through $(-1, 0)$, since the x-intercept is -1.
   Then, using the point/slope formula: $y - 0 = -\frac{5}{2}(x+1)$ or $y = -\frac{5}{2}x - \frac{5}{2}$

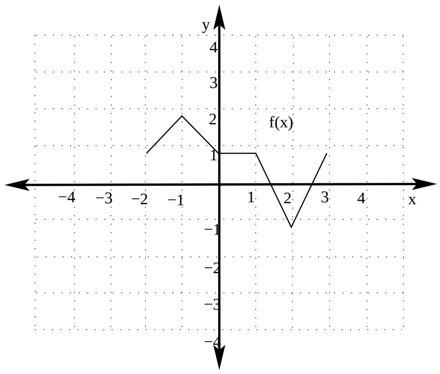
4. A 16oz jar of peanut butter cost \$1.78 in 1995. In 2005, a similar jar cost \$2.99.

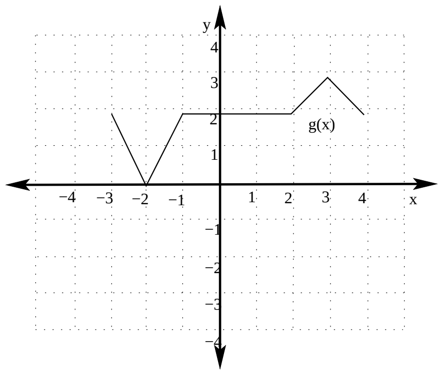
   (a) Find a line that models the price of peanut butter over time (hint: you can take $x = 0$ to represent 1995)
   Using the points $(0, 1.78)$ and $(10, 2.99)$, we find $m = \frac{2.99-1.78}{10-0} = .121$ and $b = 1.78$.
   Therefore, the line modeling the price of peanut butter is given by: $y = .121x + 1.78$, where $x = 0$ corresponds to the year 1995.

   (b) Use your model to predict the price of peanut butter in 2010.
   2010 corresponds to $x = 2010 - 1995 = 15$, and so $y = .121(15) + 1.78 = \$3.595$, or around \$3.60.

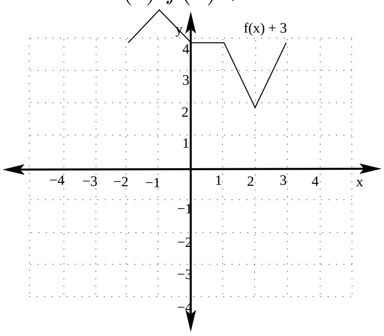
   (c) According to your model, when will the price of peanut butter reach \$5.00 for a 16oz jar?
   If $y = \$5.00$, then $5 = .121x + 1.78$, so $5 - 1.78 = .121x$, or $3.22 = .121x$
   Therefore, $x = \frac{3.22}{.121} = 26.61$.
   Hence, according to this model, the price of peanut butter will reach \$5 per 16 oz jar 26.61 years after 1995, or sometime during 2022.

5. Given the graphs of $f(x)$ and $g(x)$ shown below, use graph transformations to graph each of the following. Label at least 3 points in your final graph.

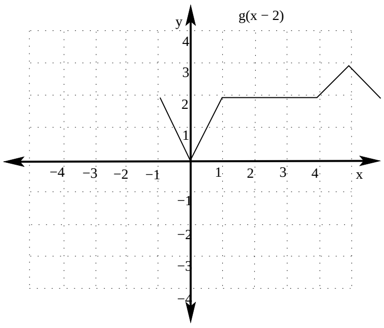




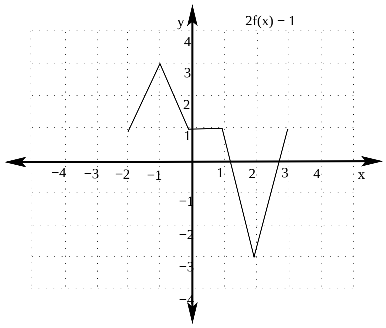
(a) $f(x) + 3$



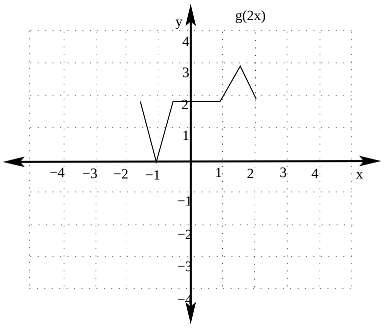
(b) $g(x - 2)$



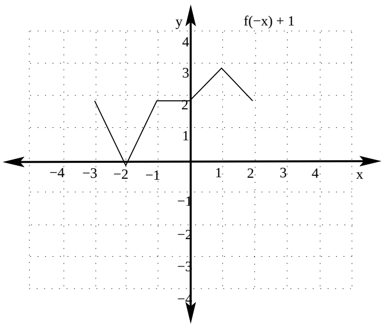
(c) $2f(x) - 1$



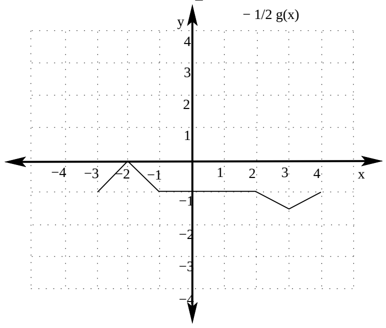
(d) $g(2x)$



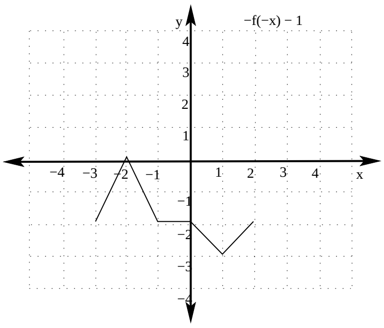
(e) $f(-x) + 1$



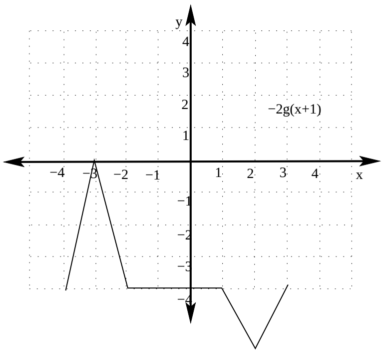
(f) $-\frac{1}{2}g(x)$



(g)c $-f(-x) - 1$



(h) $-2g(x + 1)$



6. Find the equation for the following circles:

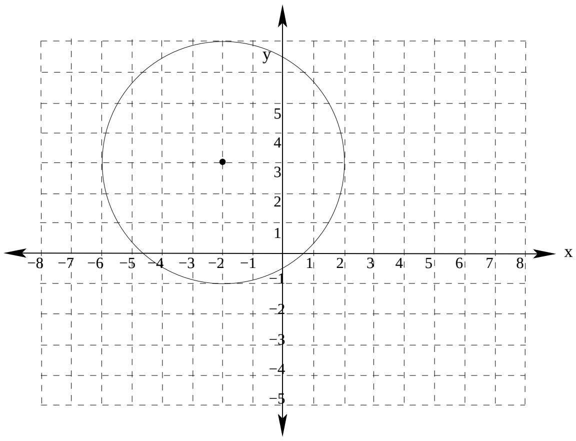
   (a) The circle with center $(4, -5)$ and radius 6
   The circle has equation $(x - 4)^2 + (y + 5)^2 = 36$

   (b) The circle with diameter passing through the points $(2, -2)$ and $(-4, -2)$
   Notice that the distance between these point is: $d(A, B) = \sqrt{(2 - (-4))^2 + (-2 - (-2))^2} = \sqrt{6^2 + 0^2} = \sqrt{36} = 6.$
   Thus the radius is *half* this distance, or $r = 3$ and the center of the circle if the midpoint of the line segment between these points, $C = \left(\frac{2+(-4)}{2}, \frac{-2+-2}{2}\right) = (-1, -2)$.
   Therefore, the circle has equation $(x + 1)^2 + (y + 2)^2 = 9$

   (c) The circle with center $(2, 1)$ and passing through the point $(5, 5)$
   Notice that the distance between these point is: $d(A, B) = \sqrt{(5 - 2)^2 + (5 - 1)^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$
   Therefore, $r = 5$ and $C = (2, 1)$, so the circle has equation $(x - 2)^2 + (y - 1)^2 = 25$

7. Graph the circle with equation $x^2 + y^2 + 4x - 6y - 3 = 0$

8. Graph the circle with equation $x^2 + y^2 + 4x - 6y - 3 = 0$
   Rearranging the terms and completing the square: $x^2 + 4x + \quad + y^2 - 6y + \quad = 3$
   Therefore, $x^2 + 4x + 4 + y^2 - 6y + 9 = 3 + 4 + 9$, or $(x + 2)^2 + (y - 3)^2 = 16$
   Thus this circle has center $(-2, 3)$ and radius $r = 4$, so the graph of the circle is:



9. Find the domain of the following functions (put your answers in interval notation):

   (a) $f(x) = \frac{x^2+x-2}{x^2-4}$
   We need to avoid making the denominator zero, so we can't have $x^2 - 4 = 0$ or $x^2 = 4$.
   Therefore, $x \neq \pm 2$.
   Therefore, in interval notation, the domain of f is: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

   (b) $f(x) = \frac{\sqrt{4-2x}}{x^2-1}$
   There are two things to worry about. First, we need the denominator to be non-zero. That is, we can't have $x^2 - 1 = 0$ or $x^2 = 1$.
   Therefore, $x \neq \pm 1$.
   Next, we can't take the square root of a negative number, so we need $4 - 2x \geq 0$.
   That is, $4 \geq 2x$, or $2 \geq x$. Combining these, the domain of f is:
   $(-\infty, -1) \cup (-1, 1) \cup (1, 2]$

   (c) $f(x) = \frac{4}{\sqrt{3x-5}}$
   Here, we need $3x - 5 > 0$, or $3x > 5$. Thus $x > \frac{5}{3}$.
   Therefore, the domain is: $(\frac{5}{3}, \infty)$

   (d) $f(x) = \frac{\sqrt{3-2x}}{2x^2+x-15}$
   There are two things to worry about. First, we need the denominator to be non-zero. That is, we can't have $2x^2 + x - 15 = 0$ or $(2x - 5)(x + 3) = 0$.
   Therefore, $x \neq \frac{5}{2}$ or $x \neq -3$.
   Next, we can't take the square root of a negative number, so we need $3 - 2x \geq 0$.
   That is, $3 \geq 2x$, or $\frac{3}{2} \geq x$. Combining these, the domain of f is:
   $(-\infty, -3) \cup (-3, \frac{3}{2}]$

10. Given that $f(x) = \sqrt{2x - 2}$ and $g(x) = \dfrac{4}{3x - 2}$

   (a) Find $\dfrac{g}{f}(3)$
   $f(3) = \sqrt{2(3) - 2} = \sqrt{4} = 2$
   $g(3) = \frac{4}{3(3)-2} = \frac{4}{7}$
   $\dfrac{g}{f}(3) = \dfrac{g(3)}{f(3)} = \dfrac{\frac{4}{7}}{2} = \dfrac{4}{7} \cdot \dfrac{1}{2} = \dfrac{2}{7}$

   (b) Find $f \circ g(2)$
   $g(2) = \frac{4}{3(2)-2} = \frac{4}{4} = 1$
   $f \circ g(2) = f(g(2)) = f(1) = \sqrt{2(1) - 2} = \sqrt{0} = 0$

11. Given that $f(x) = \sqrt{3x - 2}$ and $g(x) = x^2 - 4$

   (a) Find $g \circ f(x)$
   $g \circ f(x) = g(f(x)) = (\sqrt{3x - 2})^2 - 4 = 3x - 2 - 4 = 3x - 6 = 3(x - 2)$

   (b) Find $f \circ g(x)$
   $f \circ g(x) = f(g(x)) = \sqrt{3(x^2 - 4) - 2} = \sqrt{3x^2 - 12 - 2} = \sqrt{3x^2 - 14}$

(c) Find the domain of $g \circ f(x)$. Give your answer in interval notation.
To find the domain of $g \circ f(x) = g(f(x))$, we first find the domain of f:
$3x - 2 \geq 0$, so $3x \geq 2$ or $x \geq \frac{2}{3}$.
Next, notice that g is never undefined.
Therefore, the domain of $g \circ f(x)$ is $[\frac{2}{3}, \infty)$

(d) Find the domain of $\dfrac{f}{g}$. Give your answer in interval notation.
To be in the domain of $\dfrac{f}{g}$, we need $f(x)$ to be defined, and $g(x)$ to be defined and non-zero.
Therefore, we need $3x - 2 \geq 0$, or $3x \geq 2$, hence $x \neq \frac{2}{3}$.
We also need $x^2 - 4 \neq 0$, or $x \neq \pm 2$
Hence the domain of $\dfrac{f}{g}$ is $[\frac{2}{3}, 2) \cup (2, \infty)$

12. An oil well off the Gulf Coast is leaking, with the leak spreading oil over the surface in the shape of a circle. At any time t, in minutes, after the beginning of the leak, the radius of the circular oil slick on the surface is $r(t) = 4t$ feet. Let $A(r) = \pi r^2$ represent the area of the circle of radius r.

(a) Find $(A \circ r)(t)$
Since $r(t) = 4t$ and $A(r) = \pi r^2$, $(A \circ r)(t) = \pi(4t)^2 = 16\pi t^2$

(b) Explain what $(A \circ r)(t)$ is in practical terms.
$(A \circ r)(t)$ gives the area of the oil as a function of time in minutes.

13. Given the tables below, find the following:

x	0	2	4	6	8
f(x)	1	5	8	4	0

x	0	2	4	6	8
g(x)	2	6	5	9	7

(a) $\left(\dfrac{f}{g}\right)(8) = \dfrac{f(8)}{g(8)} = \dfrac{0}{7} = 0$

(b) $(f \circ g)(2) = f(g(2)) = f(6) = 4$

(c) $(g \circ g)(2) = g(g(2)) = g(6) = 9$

(d) $f^{-1}(5) = 2$

(e) $f(g^{-1}(9)) = f(6) = 4$

14. Determine whether or not the following functions are one-to-one. You must justify your answer to each part.

(a) $f(x) = 3x - 5$
Suppose $f(a) = f(b)$. Then $3a - 5 = 3b - 5$. Then, adding 5 to both sides of the equation:
$3a = 3b$, or, dividing both sides by 3, $a = b$
Therefore $f(x)$ is one-to-one.

(b) $f(x) = x^3 - x$
Notice that if $x^3 - x = 0$, then $x(x^2 - 1) = 0$, or $x(x-1)(x+1) = 0$. Thus $x = 0, 1, -1$
That is, $f(0) = f(1) = f(-1) = 0$. Hence $f(x)$ is *not* one-to-one.

(c) $f(x) = 3|x| - 2$
Notice that $f(2) = 3|2| - 2 = 6 - 2 = 4$, and $f(-2) = 3|-2| - 2 = 3(2) - 2 = 4$, while $2 \neq -2$. Therefore, f is *not* one-to-one.

(d) $g(x) = -\dfrac{1}{2x}$
Suppose $g(a) = g(b)$. Then $\frac{1}{2a} = \frac{1}{2b}$. But then, multiplying both sides by $(2ab)$:
$\frac{2ab}{2a} = \frac{2ab}{2b}$, or, reducing, $b = a$.
Therefore g is one-to-one.

15. Use algebra to find the inverse of each of the following functions:

(a) $f(x) = 5x - 4$

To find the inverse of f, we first solve $y = 5x - 4$ for x. To do so, we add 4 to both sides:
$y + 4 = 5x$, or, dividing both sides by 5:
$\frac{y+4}{5} = x$, or $x = \frac{y}{5} + \frac{4}{5}$.
Therefore, $f^{-1}(x) = \frac{x}{5} + \frac{4}{5}$.

(b) $f(x) = \sqrt{x - 4}$

To find the inverse of f, we first solve $y = \sqrt{x - 4}$ for x.
Squaring both sides, $y^2 = x - 4$, or, adding 4 to both sides, $y^2 + 4 = x$
Thus $f^{-1}(x) = x^2 + 4$. (Note that this inverse function is only valid on the restricted domain $x \geq 0$)

(c) $f(x) = \dfrac{5x}{3 - x}$

To find the inverse of f, we first solve $y = \frac{5x}{3-x}$ for x.
First we multiply to clear the denominator, yielding $y(3 - x) = 5x$, or $3y - xy = 5x$.
Next, we get everything involving x on one side: $3y = 5x + xy$
Then, we factor out x: $3y = x(5 + y)$, or $\frac{3y}{5+y} = x$
Therefore, exchanging x and y, we have $f^{-1}(x) = \frac{3x}{5+x}$

(d) $f(x) = \dfrac{2x - 3}{3x + 4}$

To find the inverse of f, we first solve $y = \frac{2x-3}{3x+4}$ for x.
First we multiply to clear the denominator, yielding $y(3x + 4) = 2x - 3$, or $3xy + 4y = 2x - 3$.
Next, we get everything involving x on one side: $4y + 3 = 2x - 3xy$
Then, we factor out x and divide: $4y + 3 = x(2 - 3y)$, or $\frac{4y+3}{2-3y} = x$
Therefore, exchanging x and y, we have $f^{-1}(x) = \frac{4x+3}{2-3x}$