Math 261
Curve Sketching Example

Let $f(t) = \dfrac{t^2}{t^2-1}$. Before we even begin, we should notice that $f(t)$ is not defined for $t = \pm 1$.

A. Find the intercepts.

y-intercept: $f(0) = \frac{0^2}{0^2-1} = \frac{0}{-1} = 0$

x-intercepts: For a fraction to be zero, its numerator must be zero (and its denominator must be non-zero)
Here, if $t^2 = 0$, then $t = 0$, so there is only one intercept, the point $(0,0)$ which is both an x-intercept and a y-intercept.

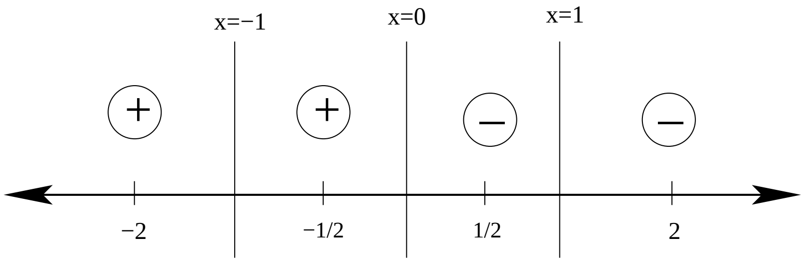
B. Finding Increasing/Decreasing Intervals and Relative Extrema Using $f'(t)$.

$$f'(t) = \frac{2t(t^2-1) - t^2(2t)}{(t^2-1)^2} = \frac{2t^3 - 2t - 2t^3}{(t^2-1)^2} = \frac{-2t}{(t^2-1)^2}$$

Critical numbers:

Notice that $f'(t)$ is *undefined* when $t^2 - 1 = 0$ or when $t = \pm 1$.
Also notice that $f'(t) = 0$ when $t = 0$.

Analyze the sign of $f'(t)$:



Therefore, $f(x)$ is increasing on the intervals: $(-\infty, -1) \cup (-1, 0)$
Similarly, $f(x)$ is decreasing on the intervals: $(0, 1) \cup (1, \infty)$

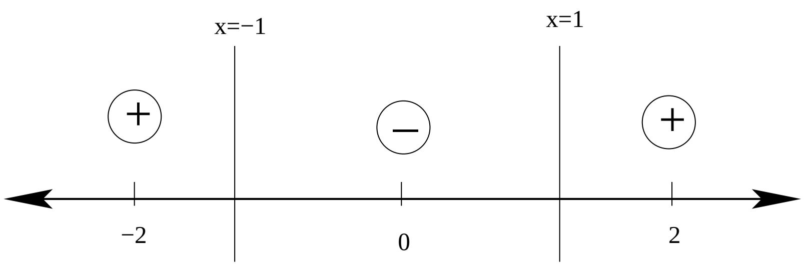
Classify Local Extrema:
Notice that $f(0)$ is defined, and $f'(t)$ goes from positive to negative at $t = 0$, so there is a local maximum when $t = 0$. The value of this maximum is $f(0) = 0$, so the local maximum occurs at the point $(0, 0)$. This is the only local extremum.

C. Find Concavity and Inflection Points Using $f''(t)$.

$$f''(t) = \frac{-2(t^2-1)^2 - (-2t)(2)(t^2-1)(2t)}{(t^2-1)^4} = \frac{(t^2-1)[-2(t^2-1) + (2t)(2)(2t)]}{(t^2-1)^4} = \frac{(t^2-1)[-2t^2+2+8t^2]}{(t^2-1)^4} = \frac{(t^2-1)(6t^2+2)}{(t^2-1)^4} = \frac{(6t}{(t^2}$$

To find the key values for the second derivative, notice that $f''(t)$ is *undefined* when $t = \pm 1$ and that $f''(t)$ is textitnever zero.

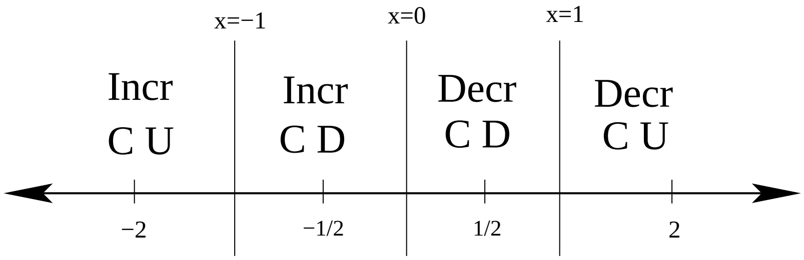
Sign testing diagram for $f''(t)$:



Therefore $f(x)$ is concave up on the intervals $(-\infty, -1) \cup (1, \infty)$ and concave down on the interval $(-1, 1)$.

Notice that there are no inflection points, since the function is undefined at $t = \pm 1$, and these are the only places where $f(t)$ changes concavity.

Combined Sign Chart:



D. Finding Asymptotes to the graph of $f(t)$:

Horizontal asymptotes:

Notice that $\lim_{x\to\infty} \frac{t^2}{t^2-1} = 1$, so $f(t)$ has horizontal asymptote $y = 1$.

Vertical asymptotes:

Notice that $\lim_{x\to -1^-} \frac{t^2}{t^2-1} = \infty$, $\lim_{x\to -1^+} \frac{t^2}{t^2-1} = -\infty$, $\lim_{x\to 1^-} \frac{t^2}{t^2-1} = -\infty$, and $\lim_{x\to 1^+} \frac{t^2}{t^2-1} = \infty$

Therefore, $f(t)$ has vertical asymptotes $t = -1$ and $t = 1$.

E. Combining All this Information to Sketch the graph of $f(t)$: