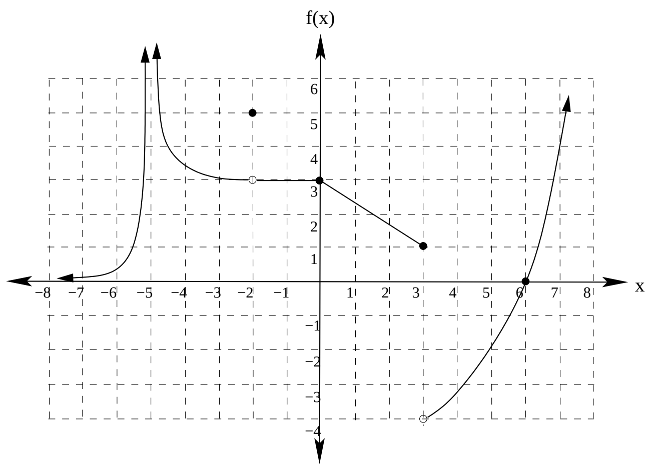
Math 261
Exam 1 Solutions
05/18/2012

Instructions: You will have 60 minutes to complete this exam. Calculators are allowed, but this is a closed book, closed notes exam. The credit given on each problem will be proportional to the amount of correct work shown. Correct answers without supporting work will receive little credit. Simplify answers when possible and follow directions carefully on each problem.

1. (2 points each) Use the information presented in the following graph to find each of the following:



(a) $f(0) = 3$

(d) $\lim_{x \to 3^-} f(x) = 1$

(g) $\lim_{x \to -5^+} f(x) = \infty$

(b) $f(2) = \frac{5}{3}$

(e) $\lim_{x \to 3^+} f(x) = -4$

(h) $\lim_{x \to -\infty} f(x) = 0$

(c) $\lim_{x \to -2} f(x) = 3$

(f) $\lim_{x \to 3} f(x)$ d.n.e.

(i) the range of $f(x) : (-4, \infty)$

(j) Find an equation for the line segment passing through $f(1)$

Notice that $m = \frac{1-3}{3-0} = \frac{2}{3}$, so the line segment is: $y = -\frac{2}{3}x + 3$ for $x \in [1, 3]$.

(k) (6 points) Give two values of x at which $f(x)$ is discontinuous and classify the type of discontinuity at each of the two values.

There are three points of discontinuity (you were asked to find and classify two, so any two of these would receive full credit):

$x = -5$ is an infinite type discontinuity.

$x = -2$ is a removable discontinuity.

$x = 3$ is a jump discontinuity.

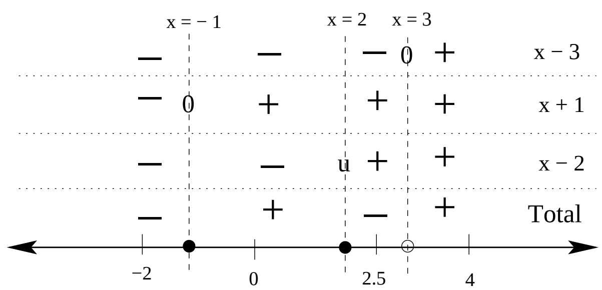
2. (2 points each) Find the exact value of each of the following:

(a) $\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ (b) $\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ (c) $\tan^{-1}(1) = \frac{\pi}{4}$

3. (6 points) Find the solution to the inequality $\dfrac{(x-3)(x+1)}{x-2} \geq 0$

The key values are: $x = -1$, $x = 2$, and $x = 3$.

Carrying out sign testing gives the following sign chart:



Therefore, the inequality has the following solution set: $[-1, 2) \cup [3, \infty)$.

4. (2 points each) Evaluate the following limits, using the symbols ∞, $-\infty$, or DNE if appropriate.
You do **not** need to justify your answers.

(a) $\lim\limits_{x\to 5} -3$ (b) $\lim\limits_{x\to 2} \dfrac{3x-7}{2x+1}$ (c) $\lim\limits_{x\to 3} \sqrt{2x-6}$

$\lim\limits_{x\to 5} -3 = -3$

$\lim\limits_{x\to 2} \dfrac{3x-7}{2x+1} = \dfrac{3(2)-7}{2(2)+1} = -\dfrac{1}{5}$

DNE, as $2x - 6 < 0$ when $x < 3$.

(d) $\lim\limits_{x\to 3^+} \dfrac{-5}{x-3}$ (e) $\lim\limits_{x\to\infty} \dfrac{7x^2-4}{5x^3+1}$ (f) $\lim\limits_{x\to\infty} \cos 2x$

$\lim\limits_{x\to 3^+} \dfrac{-5}{x-3} = -\infty$

$\lim\limits_{x\to\infty} \dfrac{7x^2-4}{5x^3+1} = \lim\limits_{x\to\infty} \dfrac{\frac{7}{x} - \frac{4}{x^3}}{5 + \frac{1}{x^3}} = \dfrac{0}{5} = 0$

DNE, since $\cos(2x)$ oscillates between -1 and 1.

5. (4 points each) Given that $\lim_{x\to 2} f(x) = 3$, $\lim_{x\to 2} g(x) = -1$, $\lim_{x\to 3} g(x) = 5$, and $\lim_{x\to 2} h(x) = \infty$, Find the following limits.

(a) $\lim_{x\to 2}[f(x) - 3g(x)]$

$= 3 - 3(-1) = 6$

(b) $\lim_{x\to 2} \frac{2g(x) + x}{x^2 f(x)}$

$= \frac{2(-1)+2}{2^2(3)} = \frac{0}{12} = 0.$

(c) $\lim_{x\to 2} \sqrt{(g \circ f)(x)}$

$= \sqrt{\lim_{x\to 2}(g(f(x)))} = \sqrt{5}.$

(d) $\lim_{x\to 2} \frac{(fg)(x)}{h(x)}$

$= \frac{\lim_{x\to 2}(fg)(x)}{\lim_{x\to 2} h(x)} = 0$

6. (6 points each) Evaluate the following limits, using the symbols ∞, $-\infty$, or DNE if appropriate. Whenever possible, use algebra to verify the value of the limit.

(a) $\lim_{x\to 9} \frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3}$

$= \lim_{x\to 9} \frac{x - 9}{(x - 9)(\sqrt{x} + 3)} = \lim_{x\to 9} \frac{1}{\sqrt{x} + 3} = \frac{1}{6}$

(b) $\lim_{x\to 1} \frac{x^2 - 1}{2x^2 - x - 1} = \lim_{x\to 1} \frac{(x - 1)(x + 1)}{(2x + 1)(x - 1)}$

$= \lim_{x\to 1} \frac{x + 1}{2x + 1} = \frac{1 + 1}{2(1) + 1} = \frac{2}{3}.$

7. (6 points) Find all vertical and horizontal asymptotes of the function $f(x) = \frac{2x^2 - x - 1}{x^3 - x}$

First, note that $f(x) = \frac{2x^2 - x - 1}{x^3 - x} = \frac{(2x + 1)(x - 1)}{x(x - 1)(x + 1)}.$

From this, we see that $f(x)$ has a removable discontinuity when $x = 1$, and infinite type discontinuities at $x = 0$ and $x = -1$.

Hence $f(x)$ has vertical asymptotes at $x = 0$ and at $x = -1$.

Next, notice that $\lim_{x\to\infty} \frac{2x^2 - x - 1}{x^3 - x} = \lim_{x\to\infty} \frac{\frac{2}{x} - \frac{1}{x^2} - \frac{1}{x^3}}{1 - \frac{1}{x^2}} = 0.$

Therefore, $f(x)$ has a horizontal asymptote: $y = 0$.

8. (7 points) Let $g(t) = \begin{cases} 2t-1 & t<3 \\ k & t=3 \\ t^2-\ell & t>3 \end{cases}$. Find values for k and ℓ so that $g(t)$ is continuous at $t=3$.

Recall that in order for $g(t)$ to be continuous at $t=3$, we must have $\lim_{t\to3^-} g(t) = \lim_{t\to3^+} g(t) = g(3)$.

Notice that $\lim_{t\to3^-} g(t) = 2(3)-1 = 5$. Then we must have $g(3)=5$, so $k=5$.

Similarly, we must have $\lim_{t\to3^+} g(t) = 3^2-\ell = 5$. Thus $9-\ell=5$, and hence $\ell = 9-5 = 4$.

9. (7 points) Given that $f(x) = x^2-1$, $\lim_{x\to2} f(x) = 3$, and $\epsilon = .001$,
find the largest δ such that if $0 < |x-2| < \delta$, then $|f(x)-3| < \epsilon$.

Suppose $|f(x)-L| < \epsilon$. Then $|x^2-1-3| = |x^2-4| < 0.001$.

Thus, $-0.001 < x^2-4 < 0.001$, or $3.999 < x^2 < 4.001$

Taking the square root of each side yields: $\sqrt{3.999} < x < \sqrt{4.001}$, or $\sqrt{3.999}-2 < x < \sqrt{4.001}-2$

Notice that $\sqrt{3.999}-2 \approx -.002501564$ while $\sqrt{4.001}-2 \approx .000249984$

Therefore, we must take $\delta \leq \sqrt{4.001}-2 \approx .000249984$.

10. (7 points) Use the formal definition of a limit to prove that $\lim_{x\to4} 3x-2 = 10$.

Let $\epsilon > 0$ and suppose that $|f(x)-L| < \epsilon$. Then $|(3x-2)-10| = |3x-12| < \epsilon$.

But then $3|x-4| < \epsilon$, so $|x-4| < \frac{\epsilon}{3}$.

Therefore, let $\delta \leq \frac{\epsilon}{3}$, and suppose $0 < |x-4| < \delta \leq \frac{\epsilon}{3}$.

Then $3|x-3| < \epsilon$.

Therefore $|3x-12| = |(3x-2)-10| < \epsilon$, or $|f(x)-L| < \epsilon$.

Thus $\lim_{x\to4} 3x-2 = 10$ $\square$.