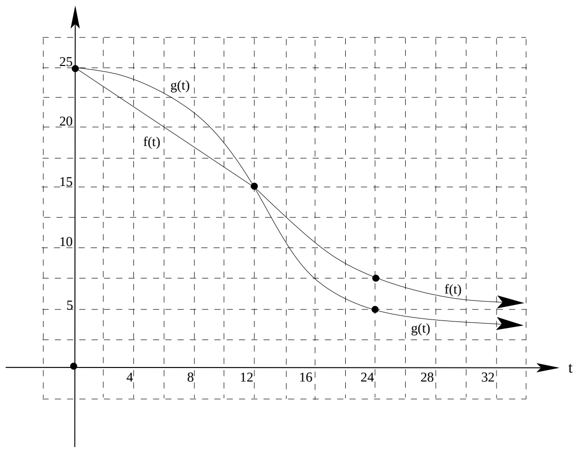
Math 261
Exam 2
05/29/2012

Name: ______________________________

Instructions: You will have 60 minutes to complete this exam. Calculators are allowed, but no other references. The credit given will be proportional to the amount of correct work shown. Correct answers without supporting work will receive little credit. Simplify answers when possible and follow directions carefully on each problem.

1. A researcher wants to compare the effects of two antibiotics. She takes two identical bacterial cultures and treats one with antibiotic A and the other with antibiotic B. Suppose $f(t)$ gives the population of culture A in millions of cells and $g(t)$ gives the population of culture B in millions of cells, where t is measured in hours (see the graph below).



(a) (3 points) How fast is the population of culture A decreasing 4 hours after it was treated with the antibiotic?

We need to find the slope of the tangent line to $f(t)$ when $t = 6$. Since this part of the graph is a line segment, we merely find the slope of the line segment using any pair of points on the segment. For example, if we take $P(0, 25)$ and $Q(12, 15)$, we see $m = \frac{25-15}{0-12} = -\frac{10}{12} = -\frac{5}{6}$ million cells per hour.

(b) (3 points) What was the average rate of change of the population of culture B from the 12th through the 24th hour after being treated?

Here, since we are computing an average rate of change, we must find the slope of the secant line between the two endpoints of the interval in question, $(12, 15)$, and $(24, 5)$.

Then $m = \frac{15-5}{12-24} = \frac{10}{-12} = -\frac{5}{6}$ million cells per hour.

(c) (3 points) Which culture's population is decreasing fastest 4 hours after the antibiotic treatments begin?

By comparing (graphically) the slopes of the tangent lines two $f(t)$ and $g(t)$ when $t = 4$, we see that $f(t)$ is steeper than $g(t)$ (in the negative direction), so Culture A is decreasing faster after 4 hours.

(d) (3 points) Which antibiotic is most effective? Justify your answer.

Looking carefully at the graph, we see that A starts out decreasing more quickly, but B is be more effective in the long term (see the righthand side of the graph). Based on this, antibiotic B is more effective.

2. (10 points) Use the definition of the derivative to find $f'(x)$ when $f(x) = \sqrt{x-3}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h-3}-\sqrt{x-3}}{h} \cdot \frac{\sqrt{x+h-3}+\sqrt{x-3}}{\sqrt{x+h-3}+\sqrt{x-3}}$$

$$= \lim_{h\to 0} \frac{x+h-3-(x-3)}{h(\sqrt{x+h-3}+\sqrt{x-3})} = \lim_{h\to 0} \frac{x+h-3-x+3}{h(\sqrt{x+h-3}+\sqrt{x-3})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h-3}+\sqrt{x-3})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h-3}+\sqrt{x-3}} = \frac{1}{\sqrt{x+(0)-3}+\sqrt{x-3}} = \frac{1}{\sqrt{x-3}+\sqrt{x-3}} = \frac{1}{2\sqrt{x-3}}$$

3. (3 points each)

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	7	3	2	4
1	0	3	-2	5
2	1	-4	0	2

Given the information in the table above and the fact that $h(x) = (g \circ f)(x)$, find each of the following.

(a) $(fg)'(2)$

Since $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

Then $(fg)'(2) = f'(2)g(2) + f(2)g'(2)$

$= (-4)(0) + (1)(2) = 0 + 2 = 2$

(b) $\left(\frac{f}{g}\right)'(2)$

Since $(f/g)'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$

Then $(f/g)'(2) = \frac{f'(2)g(2)-f(2)g'(2)}{[g(2)]^2}$

$\frac{(-4)(0)-(1)(2)}{[0]^2}$ which is undefined

(c) $h(2)$

Since $h(x) = g(f(x))$

$h(2) = g(f(2)) = g(1) = -2$

(d) $h'(2)$

Since $h(x) = g(f(x))$, $h'(x) = g'(f(x))f'(x)$

$h'(2) = g'(f(2))f'(2) = g'(1)f'(2) = (5)(-4) = -20$

4. (8 points) Find the equation of the tangent line to the graph of $y = \tan^2(x)$ when $x = \frac{\pi}{4}$.

First, notice that when $x = \frac{\pi}{4}$, $y = \tan^2(\frac{\pi}{4}) = (1)^2 = 1$

Next, $y' = 2\tan(x)\sec^2(x)$, so when $x = \frac{\pi}{4}$, $y' = 2\tan(\frac{\pi}{4})\sec^2(\frac{\pi}{4}) = \frac{2\tan(\frac{\pi}{4})}{\cos^2(\frac{\pi}{4})} = \frac{2(1)}{[\frac{\sqrt{2}}{2}]^2} = \frac{2}{\frac{1}{2}} = 4$.

Then, applying point/slope with $P(\frac{\pi}{4}, 1)$ and $m = 4$, the tangent line to this graph when $x = \frac{\pi}{4}$ is given by:

$y - 1 = 4(x - \frac{\pi}{4})$, or $y = 4x - \pi + 1$.

5. Given the relation $y^3 - 4x^2y^2 = 0$:

(a) (6 points) Find y' by implicitly differentiating $y^3 - 4x^2y^2 = 0$

Differentiating, we get $3y^2y' - 8xy^2 - 4x^2(2yy') = 0$, or $3y^2y' - 8xy^2 - 8x^2yy' = 0$.

Then $3y^2y' - 8x^2yy' = 8xy^2$, or $y'(3y^2 - 8x^2y) = 8xy^2$.

Thus $y' = \dfrac{8xy^2}{3y^2 - 8x^2y} = \dfrac{8xy}{3y - 8x^2}$

(b) (6 points) Find the equation for the tangent line to the curve at the point $(1, 4)$.

Notice that when $x = 1$, the original implicit equation becomes: $y^3 - 4y^2 = 0$. If $y = 4$, then $4^3 - 4(4^4) = 64 - 64 = 0$, so $(1, 4)$ is on this curve.

Next, using the equation above, $y' = \frac{8(1)(4)}{3(4)-8(1)^2} = \frac{32}{12-8} = \frac{32}{4} = 8$.

Therefore, the equation for the tangent line is given by: $y - 4 = 8(x - 1)$, or $y = 8x - 4$.

6. (8 points) Use the quotient rule to derive the standard formula for the derivative of $\sec(x)$. $\left[\text{Hint: } \sec x = \frac{1}{\cos x}\right]$

By the quotient rule, $\frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{0-(1)(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\frac{\sin x}{\cos x} = \sec x \tan x$.

7. (8 points) Use differentials to approximate $\sqrt{17}$. How good is your approximation?

Let $f(x) = \sqrt{x}$, take $x_0 = 16$, and $\Delta x = 1$

Then $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$.

Therefore, $f(17) \approx f(16) + f'(16)\Delta x = \sqrt{16} + \frac{1}{2\sqrt{16}} = 4 + \frac{1}{8} = 4.125$

Using a calculator, $\sqrt{17} \approx 4.123105626$, so our approximation is to within about 2 thousandths.

8. (8 points each) Find the derivative of each of the following functions. You **do not** need to simplify your answers.

   (a) $f(x) = \dfrac{\cos^4(x)}{\sqrt{x^2+1}}$

   $f'(x) = \frac{4\cos^3(x)(-\sin x)\sqrt{x^2+1} - \cos^4(x)\frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)}{\left(\sqrt{x^2+1}\right)^2}$

   $= \frac{-4\cos^3(x)\sin(x)\sqrt{x^2+1} - x\cos^4(x)(x^2+1)^{-\frac{1}{2}}}{x^2+1}$

   (b) $g(x) = x^2\sec^3(2x)$

   $g'(x) = 2x\sec^3(2x) + x^2(3\sec^2(2x))(\sec(2x)\tan(2x))(2) = 2x\sec^3(2x) + 6x^2\sec^3(2x)\tan(2x)$

9. (8 points) Find y'' if $y = \dfrac{2x+1}{1-3x}$.

   First, using the quotient rule, $y' = \dfrac{2(1-3x)-(2x+1)(-3)}{(1-3x)^2} = \dfrac{2-6x+6x+3}{(1-3x)^2} = \dfrac{5}{(1-3x)^2}$

   Then $y'' = (5)(-2)(1-3x)^{-}3(-3) = \dfrac{30}{(1-3x)^3}$

10. (8 points) In calm waters, oil spilling from the ruptured hull of a tanker spreads in all directions. If the area polluted by the spill is a circle and its radius is increasing at a rate of 2 ft/sec, how fast is the area increasing when the radius of the spill is 40 feet? (Be sure to include units in your answer.)

    Notice that the area of the oil spill is given by $V = \pi r^2$, so, differentiating implicitly:

    $\frac{dV}{dt} = 2\pi r\frac{dr}{dt} = 2\pi(40ft)(2ft/sec.) = 160\pi ft^2\,\text{sec} \approx 502.655\text{ft}^2/\text{sec}.$

    Therefore, when the radius of the spill is 40 feet, the area of the spill in increasing at a rate of approximately 502.655 square feet per second.

11. (5 points) Draw the graph of a function that is *continuous* at $x = 2$ but **not** *differentiable* at $x = 2$.