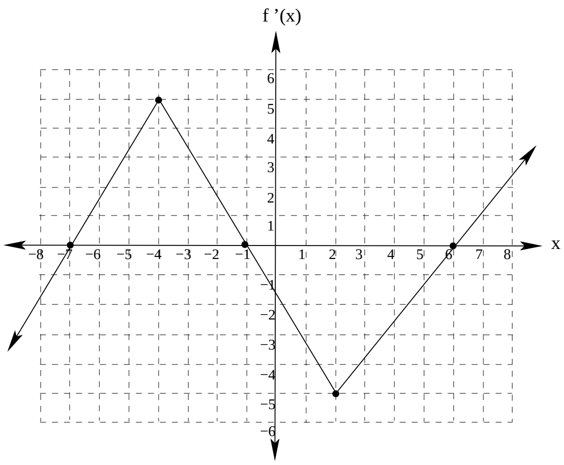
Math 261
Exam 3
06/05/2012

Name:________________________________

Instructions: You will have 50 minutes to complete this exam. Calculators are allowed, but this is a closed book, closed notes exam. The credit given on each problem will be proportional to the amount of correct work shown. Correct answers without supporting work will receive little credit. Simplify answers when possible and follow directions carefully on each problem.

1. Answer the questions below based on the graph of $f'(x)$ [the *derivative* of $f(x)$] shown here:



(a) (5 points) Find the intervals where $f(x)$ is increasing.

Notice that $f(x)$ is increasing when $f'(x) > 0$, that is, when $(-7,-1) \cup (6,\infty)$

(b) (5 points) Give the x coordinates of the local extrema of $f(x)$, and state whether each is a local maximum or a local minimum.

Notice that $f(x)$ has a local maximum whenever $f'(x)$ goes from positive to negative and $f(x)$ has a local minimum whenever $f'(x)$ goes from negative to positive. Therefore, there is a local maximum when $x = -1$, and local minima when $x = -7$ and $x = 6$.

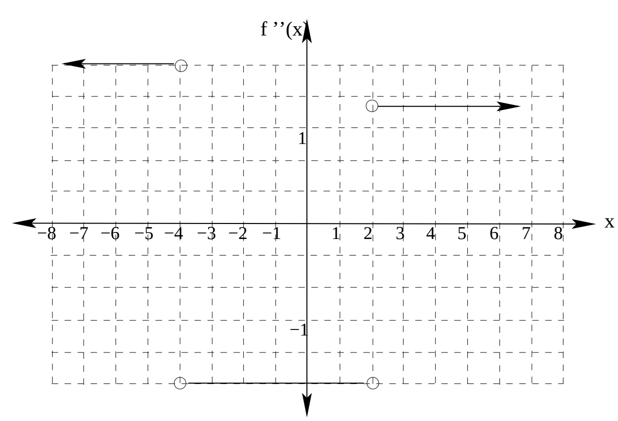
(c) (5 points) Find the intervals where $f(x)$ is concave up.

Notice that $f(x)$ is concave up whenever $f'(x)$ is increasing and $f(x)$ is concave down whenever $f'(x)$ is decreasing. Therefore $f(x)$ is concave up on the intervals: $(-\infty,-4) \cup (2,\infty)$

(d) (5 points) Give the x-coordinates of any inflection points.

Notice that $f(x)$ has inflection points whenever it changes concavity. That is, whenever $f'(x)$ goes from increasing to decreasing or from decreasing to increasing. Therefore, $f(x)$ has inflection points when $x = -4$ and $x = 2$

(e) (5 points) Draw the graph of $f''(x)$ on the grid provided:



2. (12 points) Find the absolute extrema of $f(x) = \sin^2 x + \cos x$ on the interval $[0, \pi]$

To find the absolute extrema of $f(x)$, we notice that since $f(x)$ is continuous, the EVT applies, so we need only check the value of $f(x)$ at the endpoints of the given interval and at each of the critical numbers of $f(x)$.

To find the critical numbers of $f(x)$ within the interval $[0, \pi]$, notice that $f'(x) = 2\sin x \cos x - \sin x$, which is never undefined, so we need only solve:

$2\sin x \cos x - \sin x = 0$, or $\sin x(2\cos x - 1) = 0$, which occurs when $\sin x = 0$, so $x = 0$ or $x = \pi$ and when $2\cos x = 1$, or when $\cos x = \frac{1}{2}$. That is, when $x = \frac{\pi}{3}$. (Notice that we have restricted our attention to solutions on $[0, \pi]$)

Next, we compute: $f(0) = 0^2 + 1 = 1$, $f(\pi) = 0^2 - 1 = -1$, and $f(\frac{\pi}{3}) = \left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2} = \frac{5}{4}$

Therefore, $f(x)$ attains an absolute maximum value of $\frac{5}{4}$ when $x = \frac{\pi}{3}$ and $f(x)$ attains an absolute minimum value of -1 when $x = \pi$.

3. (4 points each) For each of the following, state whether there is a local maximum, local minimum, neither, or more information is needed:

(a) $g(0) = -5$, $g'(0) = 0$, and $g''(0) = 3$

Since $g'(0) = 0$, $x = 0$ is a critical number for g. Since $g''(0) > 0$, $f(x)$ is concave up when $x = 0$. Therefore, by the second derivative test, there must be a local minimum when $x = 0$.

(b) $h(1) = 0$, $h'(1) = 1$, and $h''(1) = 0$

Since $h'(1)$ is defined and $\neq 0$, $x = 1$ is **not** a critical number for h. Therefore there is neither a local minimum nor a local maximum when $x = 1$ (notice that I did not ask about inflection points in this question).

4. (12 points) On the grid provided below, sketch the graph of a function $f(x)$ that satisfies the following:

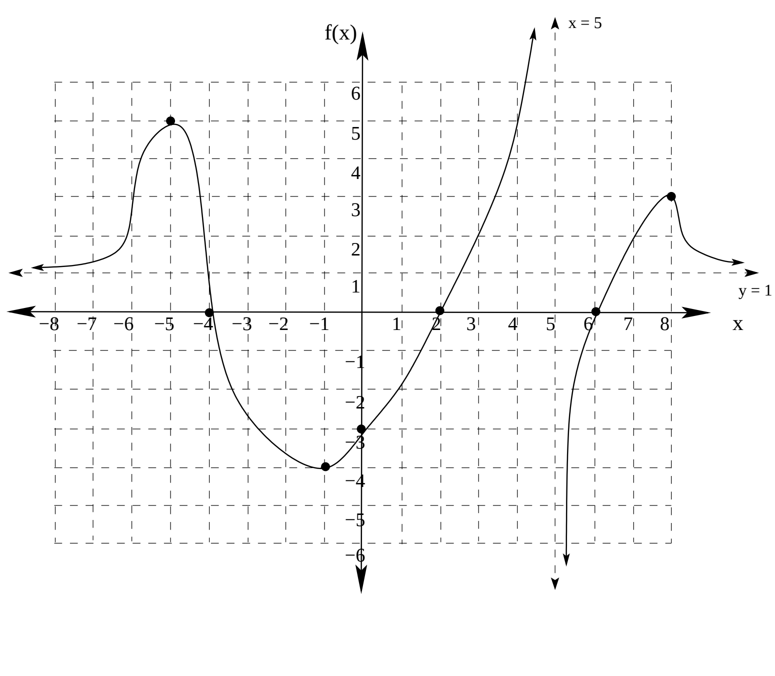
Domain: $(-\infty, 5) \cup (5, \infty)$; x-intercepts: $(-4, 0), (2, 0)$, and $(6, 0)$; y-intercept: $(0, -3)$

$f'(x) > 0$ on: $(-\infty, -5) \cup (-1, 5) \cup (5, 8)$; $f'(x) < 0$ on: $(-5, -1) \cup (8, \infty)$

$f''(x) > 0$ on: $(-\infty, -6) \cup (-4, 5) \cup (8, \infty)$; $f''(x) < 0$ on: $(-6, -4) \cup (5, 8)$

Local Max: $(-5, 5)$ and $(8, 3)$; Local Min: $(-1, -4)$

$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 1$



5. (a) (8 points) Let $f(x) = x^4 - 2x^2 - 5$. Find all values c satisfying the conclusion of the Mean Value Theorem on the interval $[-3, 3]$

Notice that $f(3) = 58$ and $f(-3) = 58$, so the slope of the secant line containing $(-3, 58)$ and $(3, 58)$ is $\frac{58-58}{3-(-3)} = \frac{0}{6} = 0$
Therefore, we are looking for all values c such that $f'(c) = 0$ within the interval $[-3, 3]$.

$f'(x) = 4x^3 - 4x = 0$, so we will look at the equation $4c^3 - 4c = 0$, or $4c(c^2 - 1) = 0$, which has solutions $c = 0$, and $c = \pm 1$
Since each of these is in $[-3, 3]$, these are the c values which satisfy the Mean Value Theorem on this interval.

(b) (5 points) Suppose that $f(2) = -1$ and $f'(x) \geq 3$ for $2 \leq x \leq 6$. What is the least possible value for $f(6)$?

Since $f'(x) \geq 3$ for $2 \leq x \leq 6$, f is continuous and differentiable on this interval.
Therefore, by the mean value theorem, $\frac{f(6)-f(2)}{6-2} = \frac{f(6)-(-1)}{4} = f'(c)$ for some $c \in (2, 6)$.
But $f'(c) \geq 3$. Thus $f(6) + 1 \geq 4 \cdot 3 = 12$, or $f(6) \geq 11$.

A simpler way to think about this is:
Since the slope of $f(x)$ is *at least* 3 throughout this interval, then as x increases from 2 to 6, $f(x)$ must increase by *at least* $4 \cdot 3 = 12$ units.

Hence $f(6) \geq -1 + 12 = 11$.

(c) (8 points) Use Newton's Method to approximate $\sqrt[3]{7}$. Use an initial guess of $x_1 = 2$ and compute x_2 and x_3. How good is your estimate?

Recall that in order to use Newton's Method, we need a function that has the value we are looking for as a root.

To accomplish this, we start with $x = \sqrt[3]{7}$, and, cubing both sides, obtain $x^3 = 7$, so we take $f(x) = x^3 - 7$.

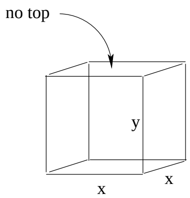
Next, we use the iterative formula: $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x+k)}$. Note that $x_1 = 2$, and $f'(x) = 3x^2$.

Then $x_2 = 2 - \dfrac{f(2)}{f'(2)} = 2 - \dfrac{1}{12} = \dfrac{23}{12}$.

Finally, $x_3 = \dfrac{23}{12} - \dfrac{f\left(\frac{23}{12}\right)}{f'\left(\frac{23}{12}\right)} \approx 1.912938458$.

Using a calculator, $\sqrt[3]{7} \approx 1.912931183$, so even after only two iterations, our estimate is quite good.

6. (13 points) A company wants to manufacture an open top box with a square base as cheaply as possible. The box is to have a volume of 108 cubic inches. Find the dimensions of the box that will minimize the amount of material used (i.e. the surface area of the box). Note that the finished box has no top!



Let x be the length of a side of the square base of the box, and y the height of the box. Then the volume of the box is given by $V = x^2y = 108$ cubic inches. Hence $y = \frac{108}{x^2}$. Next, since the box has no top, the surface area of the box is given by the area of the base plus the sum of the area of the four sides. That is, $A = x^2 + 4xy$. Substituting, $A(x) = x^2 + (4x) \cdot \frac{108}{x^2} = x^2 + \frac{432}{x}$.

Now, we find the critical numbers for this function: $A'(x) = 2x - 432x^{-2} = 2x - \frac{432}{x^2} = \frac{2x^3 - 432}{x^2}$. Notice that $x = 0$ does not make sense for a box, so we only consider $2x^3 - 432 = 0$, in which case, $2x^3 = 432$, $x^3 = 216$, so $x = \sqrt[3]{216} = 6$ is a critical number.

Notice that $A'(1) = -430 < 0$, and $A'(8) = \frac{1024-432}{64} = \frac{592}{64} = \frac{37}{4} > 0$, so there is a local minimum at $x = 6$. Since x could hypothetically be as large as we wish, there are no other boundary points. Hence the surface area of the box must be minimized when $x = 6$ and $y = \frac{108}{36} = 3$.

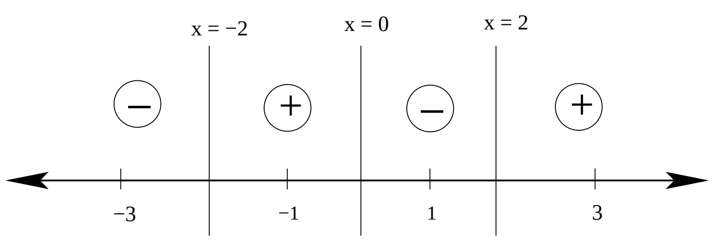
Thus the dimensions of the box with minimal surface area and volume 108 cubic inches is 6 inches long, 6 inches wide, and 3 inches tall.

7. Given the **derivative** of f: $f'(x) = \dfrac{2x}{x^2 - 4}$.

   (a) (6 points) Find the intervals where $f(x)$ is increasing.

   Notice that $f'(x) = 0$ when $x = 0$ and $f'(x)$ is undefined when $x^2 - 4 = 0$, or when $x = \pm 2$.

   If we use test points to check the sign of $f'(x)$ on the intervals between these critical numbers:



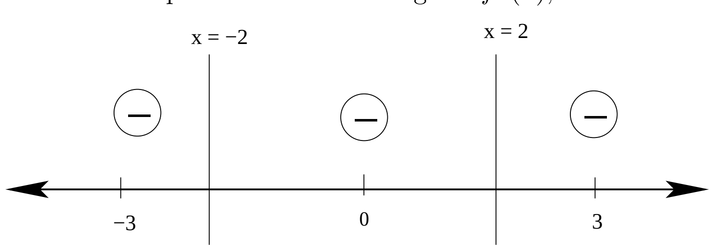
   Then we see that $f(x)$ is increasing on the intervals: $(-2, 0) \cup (2, \infty)$

   (b) (7 points) Find the intervals where $f(x)$ is concave down.

   Differentiating, we see that $f''(x) = \frac{(2)(x^2-4)-(2x)(2x)}{(x^2-4)^2} = \frac{2x^2-8-4x^2}{(x^2-4)^2} = \frac{-2x^2-8}{(x^2-4)^2} = -\frac{2(x^2+4)}{(x^2-4)^2}$,

   which is undefined when $x = \pm 2$ and is never equal to zero.

   If we use test points to check the sign of $f''(x)$, we see:



   Therefore $f(x)$ is concave down on the intervals: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$