Math 261
Exam 4 - Practice Problem
Solutions

1. Suppose you throw a ball vertically upward. If you release the ball 7 feet above the ground at an initial speed of 48 feet per second, how high will the ball travel? (Assume gravity is $-32ft/sec^2$)

   We know that $a(t) = -32$, $v(0) = 48$, and $s(0) = 7$.

   Antidifferentiating, $v(t) = -32t + C$, so $v(0) = 48 = -32(0) + C$, so $C = 48$, and $v(t) = -32t + 48$.

   Antidifferentiating again, $s(t) = -16t^2 + 48t + D$, so $s(0) = 7 = D$, and $s(t) = -16t^2 + 48t + 7$.

   The max height ocurs when $v(t) = 0$, that is, when $-32t + 48 = 0$, or $32t = 48$, so when $t = \frac{48}{32} = \frac{3}{2}$. [Notice $s''(t) = a(t) < 0$, so we know it is a maximum]

   Thus, the max height is: $s(\frac{3}{2}) = -16\left(\frac{3}{2}\right)^2 + 48\left(\frac{3}{2}\right) + 7 = 43$, or 43 feet.

2. Find each of the following indefinite integrals:

   (a) $\displaystyle\int \frac{x^{\frac{3}{2}} - 7x^{\frac{1}{2}} + 3}{x^{\frac{1}{2}}} dx$

   $\displaystyle= \int x - 7 + 3x^{-\frac{1}{2}} dx = \frac{1}{2}x^2 - 7x + 6x^{\frac{1}{2}} + C$

   (b) $\displaystyle\int \sin^3 x \cos x dx$

   Let $u = \sin x$. Then $du = \cos x\, dx$, and we have $\displaystyle\int u^3\, du = \frac{1}{4}u^4 + C$.

   Thus the indefinite integral is: $\frac{1}{4}\sin^4 x + C$.

   (c) $\displaystyle\int 5x(x^2 + 1)^8 dx$

   Let $u = x^2 + 1$. Then $du = 2x\, dx$, or $\frac{1}{2}\, du = dx$, so we have $\displaystyle\int \frac{5}{2}u^8 du = \frac{5}{2}\frac{1}{9}u^9 + C = \frac{5}{18}u^9$.

   Thus the indefinite integral is: $\frac{5}{18}(x^2 + 1)^9 + C$.

   (d) $\displaystyle\int \frac{x}{\sqrt{x+1}} dx$

   This is a trickier substituion problem. Let $u = x + 1$. Then $u - 1 = x$, and $du = dx$.

   Thus we have the indefinite integral: $\displaystyle\int \frac{u-1}{\sqrt{u}}\, du = \int u^{\frac{1}{2}} - u^{-\frac{1}{2}} du = \frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C$.

   Thus the indefinite integral is: $\frac{2}{3}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C$.

3. Solve the following differential equations under the given initial conditions:

   (a) $\frac{dy}{dx} = \sin x + x^2; y = 5$ when $x = 0$

   Antidifferentiating, $y = -\cos x + \frac{1}{3}x^3 + C$.

   Therefore, $5 = -\cos(0) + \frac{1}{3}(0)^3 + C$, or $5 = -1 + 0 + C$, so $C = 6$.

   Therefore, $y = -\cos x + \frac{1}{3}x^3 + 6$.

(b) $g''(x) = 4\sin(2x) - \cos(x); g'(\frac{\pi}{2}) = 3; g(\frac{\pi}{2}) = 6$

Antidifferentiating, $g'(x) = -2\cos(2x) - \sin x + C$

Therefore, $3 = -2\cos(\pi) - \sin\frac{\pi}{2} + C = -2(-1) - 1 + C$, so $3 = 2 - 1 + C$, or $C = 2$.

Hence $g'(x) = -2\cos(2x) - \sin x + 2$. But then $g(x) = -\sin(2x) + \cos x + 2x + D$.

Moreover, $6 = -\sin(\pi) + \cos\frac{\pi}{2} + 2\left(\frac{\pi}{2}\right) + D$, or $6 = 0 + 0 + \pi + D$, so $D = 6 - \pi$

Thus $g(x) = -\sin(2x) + \cos x + 2x + 6 - \pi$.

4. Express the following in summation notation:

(a) $2 + 5 + 10 + 17 + 26 + 37 = \sum_{k=1}^{6} \left(k^2 + 1\right)$

(b) $x^2 + \frac{x^3}{4} + \frac{x^4}{9} + ... + \frac{x^{11}}{100} = \sum_{k=1}^{10} \frac{x^{k+1}}{k^2}$

5. Evaluate the following sums:

(a) $\sum_{k=2}^{5} k^2(k+1)$

$= \sum_{k=2}^{5} k^3 + k^2 = (8+4) + (27+9) + (64+16) + (125+25) = 278$

(b) $\sum_{k=3}^{20} k^3 - k^2$

$= \sum_{k=1}^{20} k^3 - \sum_{k=1}^{20} k^2 - \sum_{k=1}^{2} k^3 + \sum_{k=1}^{2} k^2$

$= \left(\frac{(20)(21)}{2}\right)^2 - \frac{(20)(21)(41)}{6} - 1 - 8 + 1 + 4 = (210)^2 - 2870 - 9 + 5 = 41,226$

6. Express the following sums in terms of n:

(a) $\sum_{k=1}^{n} 3k^2 - 2k + 10$

$= 3\sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 10 = 3\frac{(n)(n+1)(2n+1)}{6} - 2\frac{(n)(n+1)}{2} + 10n$

$= n^3 + \frac{3n^2}{2} + \frac{n}{2} - n^2 - n + 10n = n^3 + \frac{n^2}{2} + \frac{19n}{2}$

(b) $\sum_{k=3}^{n} k(k^2 - 1)$

$= \sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} k - \left(\sum_{k=1}^{2} k^3 - \sum_{k=1}^{2} k\right)$

$= \left(\frac{(n)(n+1)}{2}\right)^2 - \frac{(n)(n+1)}{2} - (1 + 8 - 1 - 2) = \frac{n^4+2n^3+n^2}{4} - \frac{n^2+n}{2} - 6 = \frac{n^4}{4} + \frac{n^3}{2} - \frac{n^2}{4} - \frac{n}{2} - 6.$

7. Consider $f(x) = 3x^2 - 5$ in the inteval $[3, 7]$

(a) Find a summation formula that gives an estimate the definite integral of f on $[3, 7]$ using n equal width rectangles and using midpoints to give the height of each rectangle. You do not have to evaluate the sum or find the exact area.

Notice that $\Delta x = \frac{7-3}{n} = \frac{4}{n}$. Since we want to use midpoints for our heights, $x_k = 3 + k\Delta x - \frac{\Delta x}{2} = 3 + \frac{4k-2}{n}$.

Therefore, $A_n = \sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n} \left[3\left(3 + \frac{4k-2}{n}\right)^2 - 5\right]\left(\frac{4}{n}\right)$

(b) Find the norm of the partition $P : 3 < 3.5 < 5 < 6 < 6.25 < 7$

The norm is the widest gap in the partition: $5 - 3.5 = 1.5$

(c) Find the approximation of the definite integral of f on $[3, 7]$ using the Riemann sum for the partition P given in part (b).

$A \approx \sum_{k=1}^{5} f(x_k)\Delta x_k = f(3.25)(.5) + f(4.25)(1.5) + f(5.5)(1) + f(6.125)(.25) + f(6.625)(.75) = (26.6875)(.5) + (49.1875)(1.5) + (85.75)(1) + (107.546875)(.25) + (126.671875)(.75) = 294.765625$

8. Assume f is continuous on $[-5, 3]$, $\int_{-5}^{-1} f(x)\ dx = -7$, $\int_{-1}^{3} f(x)\ dx = 4$, and $\int_{1}^{3} f(x)\ dx = 2$. Find:

(a) $\int_{3}^{-1} f(x)\ dx = -\int_{-1}^{3} f(x)\ dx = -4$

(b) $\int_{-5}^{1} f(x)\ dx = \int_{-5}^{-1} f(x)\ dx + \int_{-1}^{3} f(x)\ dx - \int_{1}^{3} f(x)\ dx = -7 + 4 - 2 = -5$

(c) $\int_{-5}^{3} f(x)\ dx = \int_{-5}^{-1} f(x)\ dx + \int_{-1}^{3} f(x)\ dx = -7 + 4 = -3$

(d) $\int_{-1}^{-1} f(x)\ dx = 0$

(e) Find the average value of f on $[-5, -1]$

$= \frac{1}{-1 - (-5)} \int_{-5}^{-1} f(x)\ dx = \frac{1}{4} \cdot (-7) = -\frac{7}{4}$

9. Evaluate the following:

(a) $\int_{1}^{4} x^3 + \frac{1}{\sqrt{x}} + 2\ dx$

$= \frac{1}{4}x^4 + 2x^{\frac{1}{2}} + 2x|_1^4 = \left[\frac{1}{4}4^4 + 2 \cdot 4^{\frac{1}{2}} + 2(4)\right] - \left[\frac{1}{4}1^4 + 2 \cdot 1^{\frac{1}{2}} + 2(1)\right] = 76 - 4.25 = 71.75$

(b) $\int_{0}^{1} x^2(2x^3 + 1)^2\ dx$

Let $u = 2x^3 + 1$. Then $du = 6x^2$, or $\frac{1}{6}du = dx$.

Notice $2(0)^3 + 1 = 1$, and $2(1)^3 + 1 = 3$

Then we have $\int_{1}^{3} \frac{1}{6}u^2\ du = \frac{1}{18}u^3|_1^3 = \frac{1}{18}(3^3 - 1^3) = \frac{26}{18} = \frac{13}{9}$.

(c) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^3(2x)\cos(2x)\ dx$

Let $u = \sin(2x)$. Then $du = 2\cos(2x)dx$ or $\frac{1}{2}du = \cos(2x)dx$

Notice that $\sin(2\frac{\pi}{6}) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, while $\sin\pi = 0$

Then we have $\displaystyle\int_{\frac{\sqrt{3}}{2}}^{0} \frac{1}{2}u^3 du = \frac{1}{4}u^4|_{\frac{\sqrt{3}}{2}}^{0} = 0 - \frac{1}{8}\cdot\left(\frac{\sqrt{3}}{2}\right)^4 = -\frac{9}{128}$

(d) $\int_{-\pi}^{\pi} \sin x\ dx = 0$, since $\sin x$ is an odd function.

(e) $\displaystyle\frac{d}{dx}\left(\int_1^3 t\sqrt{t^2-1}\right)\ dt = 0$, since a definite integral gives a constant, and the derivative of a constant is zero.

(f) $\displaystyle\int_1^3 \left[\frac{d}{dx}\left(t\sqrt{t^2-1}\right)\ dt\right] = (3\sqrt{3^2-1}) - (1\sqrt{1^2-1}) = 3\sqrt{8}$

10. Suppose $\displaystyle G(x) = \int_2^x \frac{1}{t^2+1}\ dt$

(a) Find $G'(2) = \frac{1}{2^2+1} = \frac{1}{5}$

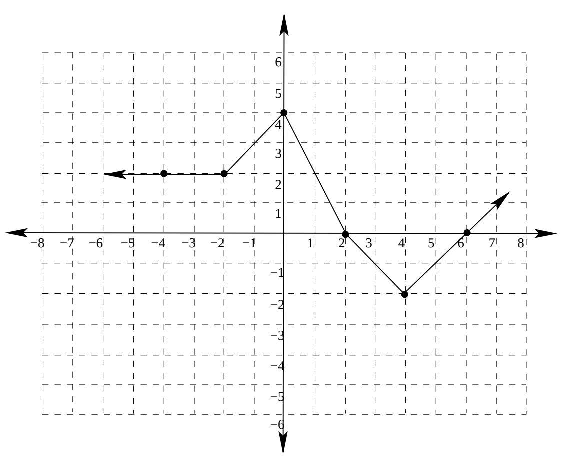
(b) Find $G'(x^2) = \frac{1}{x^4+1}$ [Note: $G'(x^2) \neq \frac{d}{dx}G(x^2)$]

(c) Find $G''(3)$

Notice that $\frac{d}{dx}\frac{1}{x^2+1} = \frac{-2x}{(x^2+1)^2}$.

Thus $G''(3) = \frac{-2(3)}{(3^2+1)^2} = \frac{-6}{(10)^2} = \frac{-3}{50}$.

11. Given the following graph of $f(x)$ and the fact that $\displaystyle G(x) = \int_{-2}^x f(t)\ dt$:



(a) Find $G(6) = \int_{-2}^6 f(t)\ dt = 2.5 + 3.5 + \frac{1}{2}(2)(4) - \frac{1}{2}(4)(2) = 6$ [Compute area directly]

(b) Find $G'(6) = f(6) = 0$

(c) Find $G''(6) = f'(6) = 1$ [Compute slope of line segment on graph at $x = 6$]

12. (a) Use the Trapeziodal Rule with $n = 4$ to approximate $\int_0^4 2x^3\,dx$

$A \approx \frac{4-0}{4 \cdot 2}[f(0)+2f(1)+2f(2)+2f(3)+f(4)] = \frac{1}{2}[0+2(2\cdot1)+2(2\cdot8)+2(2\cdot27)+2\cdot64] = \frac{1}{2}[272] = 136$

(b) Find the maximum possible error in your approximation from part (a).

Notice that $f'(x) = 6x^2$ and $f''(x) = 12x$. This has a maximum of 48 when $x = 4$ on the interval $[0, 4]$.

Therefore, Error $\leq \frac{M(b-a)^3}{12n^2} = \frac{48(4)^3}{12\cdot4^2} = 16$.

(c) Use the Fundamental Theorem of Calculus to find $\int_0^4 2x^3\,dx$ exactly. How far off was your estimate? How does the actual error compare to the maximum possible error?

$\int_0^4 2x^3\,dx = \frac{1}{2}x^4|_0^4 = \frac{1}{2}4^4 - 0 = 128$

The actual error is 136 - 128 = 8 (only half of the maxumum possible error for $n = 4$).

(d) Determine the minimum number of rectangles should be used in order to guarantee an approximation of $\int_0^4 (2x^3)\,dx$ is accurate to within .0005 when using the Trapezoid Rule.

From above, since $f'(x) = 6x^2$, then $f''(x) = 12x$, which has a maximum of 48 when $x = 4$ on the interval $[0, 4]$. Therefore, we want the error of a trapezoid approximation using n equal width trapezoids to satisfy:

Error $\leq \frac{M(b-a)^3}{12n^2} = \frac{48(4)^3}{12\cdot n^2} \leq .0005$.

So we need $48(4)^3 \leq .0005(12)n^2$, or $3072 \leq .006n^2$.

Thus $512,000 \leq n^2$, or $715.54 \leq n$, so to be sure to have an estimate within .0005, we would need to take $n = 716$. That is, we would need to use 716 trapezoids.

(In contrast, notice that Simpson's rule would give the exact area using $n = 4$.)