Math 261
Exam 4 Solutions **Name:**____________________________

Instructions: You will have 60 minutes to complete this exam. Calculators are allowed, but this is a closed book, closed notes exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit. Simplify answers when possible and follow directions carefully on each problem.

1. Let $f(x) = 4 - x^2$.

   (a) (4 points) *Approximate* the area under $f(x)$ on $[0, 2]$ using two rectangles and using right hand endpoints.

   First, notice that since $a = 0$, $b = 2$, and $n = 2$, then $\Delta x = \frac{b-a}{n} = \frac{2-0}{2} = 1$. Since we are using right-hand endpoints, $x_1 = 1$ and $x_2 = 2$.

   Therefore, $A \approx \displaystyle\sum_{k=1}^{2} f(x_k)\Delta x = f(1)(1) + f(2)(1) = 3(1) + (0)(1) = 3\,\text{units}^2$.

   (b) (5 points) Use summation notation to write an expression that represents approximating the area under $f(x)$ on $[0, 2]$ using n rectangles.

   In a general regular right-handed sum, $\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$ and $x_k = 0 + k\Delta x = \frac{2k}{n}$.

   $$\text{Then } A \approx \sum_{k=1}^{n} f\left(\frac{2k}{n}\right)\left(\frac{2}{n}\right) = \sum_{k=1}^{n}\left[4 - \left(\frac{2k}{n}\right)^2\right]\left(\frac{2}{n}\right) = \sum_{k=1}^{n}\left(4 - \frac{4k^2}{n^2}\right)\left(\frac{2}{n}\right) = \sum_{k=1}^{n}\frac{8}{n} - \frac{8k^2}{n^3}$$

   (c) (7 points) Use the expression you found in part (b) to find the *exact area* under $f(x)$ on $[0, 2]$ by first using summation formulas and then taking the limit as $n \to \infty$.

   $$\text{Using the result from above, } A = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{8}{n} - \frac{8k^2}{n^3} = \lim_{n\to\infty}\frac{8}{n}\sum_{k=1}^{n}1 - \frac{8}{n^3}\sum_{k=1}^{n}k^2$$

   $$= \lim_{n\to\infty}\frac{8}{n}(n) - \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = \lim_{n\to\infty} 8 - \frac{8}{n^2}\left(\frac{2n^3+3n^2+n}{6}\right)$$

   $$= \lim_{n\to\infty} 8 - \left(\frac{16n^3+24n^2+8n}{6n^3}\right) = \lim_{n\to\infty} 8 - \left(\frac{16+\frac{24}{n}+\frac{8}{n^2}}{6}\right) = 8 - \frac{16}{6} = \frac{24}{3} - \frac{8}{3} = \frac{16}{3}\,\text{units}^2.$$

   (d) (4 points) Verify your answer to part (c) by using the Fundamental Theorem of Calculus to evaluate $\displaystyle\int_0^2 f(x)\,dx$.

   $$\int_0^2 (4 - x^2)\,dx = 4x - \frac{1}{3}x^3\Big|_0^2 = (4)(2) - \frac{1}{3}(2)^2 - (0-0) = 8 - \frac{8}{3} = \frac{24-8}{3} = \frac{16}{3}\,\text{units}^2.$$

2. (4 points each) Compute the following:

(a) $\displaystyle\int_0^2 \frac{d}{dt}\left(\frac{1}{t^3+4}\right)\,dt$

$\displaystyle = \frac{1}{2^3+4} - \frac{1}{0^3+4}$

$\displaystyle = \frac{1}{12} - \frac{1}{4} = \frac{1-3}{12} = -\frac{1}{6}$

(b) $\displaystyle\frac{d}{dt}\left[\int_0^2 \frac{1}{t^3+4}\,dt\right]$

$\displaystyle = \frac{d}{dt}\left(F(2) - F(0)\right)$

$= 0$ (Notice that we are differentiating a constant.)

(c) $\displaystyle\frac{d}{dx}\left(\int_0^{x^2} \frac{1}{t^3+4}\,dt\right)$

$$\frac{d}{dx}\left[F(x^2) - F(0)\right] = F'(x^2)(2x) - 0 = \frac{1}{(x^2)^3+4}(2x) = \frac{2x}{x^6+4}$$

3. (4 points each) Suppose that $f(x)$ is a continuous function satisfying: $\displaystyle\int_{-2}^{5} f(x)\,dx = 11$, $\displaystyle\int_{5}^{7} f(x)\,dx = -4$, and $\displaystyle\int_{5}^{7} g(x)\,dx = -2$.

Find:

(a) $\displaystyle\int_{-2}^{7} f(x)\,dx$

$= 11 - 4 = 7$

(b) $\displaystyle\int_{5}^{7} 2g(x) - f(x)\,dx$

$= 2(-2) - (-4) = -4 + 4 = 0$

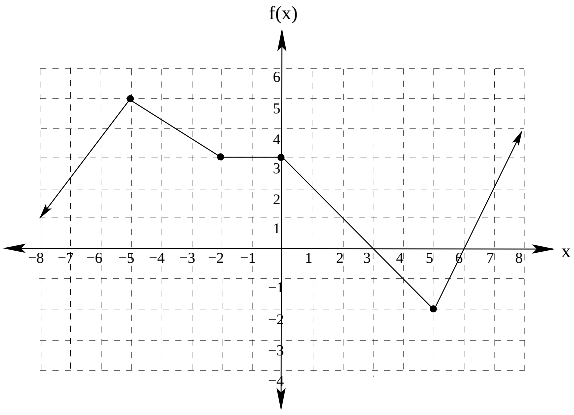
(c) $\displaystyle\int_{7}^{5} f(x)\,dx$

$= -(-4) = 4$

(d) the average value of $g(x)$ on $[5, 7]$

$= \frac{1}{7-5}\int_5^7 g(x), dx = \frac{1}{2}(-2) = -1$

4. (4 points each) Given the following graph of $f(x)$, let $g(x) = \int_{-2}^{x} f(t)\,dt$.



(a) (4 points) Find $g(2)$.

$= 2(3) + \left(\frac{3+1}{2}\right) \cdot 2$

$= 6 + (2)(2) = 10$

(b) (4 points) Find $g'(2)$

$g'(2) = f(2) = 1$

(c) (4 points) Find $g''(2)$

$g''(2) = f'(2) = -1$

5. (7 points each) Evaluate each of the following:

(a) $\int 3x\sqrt{4x^2-1}\,dx$

Let $u = 4x^2 - 1$. Then $du = 8x\,dx$, so $\frac{3}{8}\,du = 3x\,dx$. Substituting, we obtain:

$$\int \frac{3}{8}u^{\frac{1}{2}}\,du = \frac{3}{8}\frac{2}{3}u^{\frac{3}{2}} + C = \frac{1}{4}\sqrt{(4x^2-1)^3} + C$$

(b) $\int \tan^3 x \sec^2 x\,dx$

Let $u = \tan x$. Then $du = sec^2 x\,dx$. Substituting, we obtain:

$$\int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}\tan^4 x + C.$$

(c) $\int_0^{\frac{\pi}{3}} 4\sin\theta\cos\theta\,d\theta$

Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$. Also, $u(0) = \sin(0) = 0$, and $u\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$. Substituting, we obtain:

$$\int_0^{\frac{\sqrt{3}}{2}} 4u\,du = 2u^2\Big|_0^{\frac{\sqrt{3}}{2}} = 2\left(\frac{\sqrt{3}}{2}\right)^2 - 2(0)^2 = 2\cdot\left(\frac{3}{4}\right) = \frac{3}{2}.$$

6. Time trials have shown that a DeLorean automobile can accelerate from 0 to 60 miles per hour in 4 seconds.

(a) (6 points) Assuming that the DeLorean undergoes constant acceleration, find functions $v(t)$ and $a(t)$ representing the velocity and acceleration of the DeLorean as a function of time. [Hint: watch your units. There are 5280 feet in 1 mile and 360 seconds in 1 hour.]

First notice that we know that the DeLorean undergoes constant acceleration, but we do not know the value of the acceleration constant. Therefore, $a(t) = k$, so, antidifferentiating, the velocity of the car is given by $v(t) = kt + C$ and the position of the car is given by $s(t) = \frac{1}{2}kt^2 + Ct + D$. Also, we know from the description of the situation that $v(0) = 0$, $v(4) = 60mph$ and $s(0) = 0$. Converting miles per hour into feet per second:
$60\frac{miles}{hour} \cdot \frac{5280ft}{1mile} \cdot \frac{1hr}{60min.} \cdot \frac{1min.}{60sec.} = 88\frac{ft.}{sec.}$
Then $v(0) = 0 = k(0) + C$, so $C = 0$, and $v(4) = 88 = 4k$, so $k = 22$.
Hence $a(t) = 22\frac{ft.}{sec^2}$ and $v(t) = 22t$ in $\frac{ft.}{sec.}$

(b) (4 points) If Marty gets into a DeLorean parked at the Twin Pines Mall and accelerates, how long will it take him to get the DeLorean up to exactly 88 mph?

Notice that $88\frac{miles}{hour} \cdot \frac{5280ft}{1mile} \cdot \frac{1hr}{60min.} \cdot \frac{1min.}{60sec.} = \frac{1936}{15}\frac{ft.}{sec.} \approx 129.0667\frac{ft.}{sec.}$
Substituting this into the velocity equation, we see that the car reaches $88mph$ when $\frac{1936}{15} = 22t$, so $t = \frac{88}{15} \approx 5.8667$, or after about 5.8667 seconds.

(c) (5 points) If there is a photo mat exactly 380 feet in front of his starting location and he drives straight toward it, will he get to 88 mph before running into the photo mat? Justify your answer.

To find out if the car has enough room to get up to $88mph$, we will check to see how much road is needed to attain that speed. That is, how much distance is covered by the car in 5.8667 seconds. From above, $s(t) = \frac{1}{2}kt^2 + Ct + D$. Recall that $C = 0$ and $k = 22$. Also, since $s(0) = 0$, $D = 0$. Therefore $s(t) = \frac{1}{2}22t^2 + (0)t + (0) = 11t^2$. Therefore, $s(5.8667) = 11(5.8667)^2 \approx 378.59$ feet. Since this distance is less than 380 feet, we can safely conclude that the DeLorean is able to get up to 88 mph before running out of room.

7. (5 points each)

(a) Use the Trapezoidal Rule with $n = 4$ to approximate $\displaystyle\int_{-1}^{3} \left(x^3 - 3x^2\right)\, dx$

Notice that $n = 4$, so $\Delta x = \frac{b-a}{n} = \frac{3-(-1)}{4} = \frac{4}{4} = 1$.
Recall that in the Trapezoidal Rule, $A \approx \frac{b-a}{2n}\left(f(a) + 2f(a+\Delta x) + ... + 2f(a+(k-1)\Delta x) + f(x_n)\right)$
$= \frac{3-(-1)}{2\cdot 4}\left(f(-1) + 2f(0) + 2f(1) + 2f(2) + f(3)\right) = \frac{1}{2}\left(-4 + 2(0) + 2(-2) + 2(-4) + 0\right)$
$= \frac{1}{2}[-16] = -8$.

(b) Find the maximum possible error for your approximation from part (a).

Recall that the maximum error for the Trapezoidal Rule is given by $Error \leq \frac{M(b-a)^3}{12n^2}$ where M is the maximum absolute value of the second derivative $f''(x)$ in the interval $[a, b]$.

In this case, $f'(x) = 3x^2 - 6x$ and $f''(x) = 6x - 6$. Since $f''(x)$ is linear with positive slope, it's extrema occur at the endpoints of the interval. Notice that $f''(-1) = -12$ and $f''(3) = 12$, so $M = 12$ and $n = 4$

$Error \leq \frac{12(4)^3}{12(4)^2} = 4$

(c) Determine the minimum number of rectangles should be used in order to guarantee an approximation of $\displaystyle\int_{-1}^{3} \left(x^3 - 3x^2\right)\, dx$ is accurate to within .001 when using the Trapezoid Rule.

We need $Error = \frac{M(b-a)^3}{12n^2} \leq 0.001$, or $\frac{12(4)^3}{12n^2} \leq 0.001$.
Then $\frac{12(4)^3}{12(0.001)} \leq n^2$, or $n^2 \geq 64,000$. Hence $n \geq \sqrt{64,000} \approx 252.98$
Therefore the minimum is $n = 253$.