Math 261

IV, EV, Rolle's and MV Theorems

The Intermediate Value Theorem: If a function f is continuous on a closed interval $[a, b]$ and w is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = w$.

• *Intuitively*, this means that a continuous function f on an closed interval $[a, b]$ attains every y value in between it boundary values ($f(a)$ and $f(b)$) at least once in the interval $[a, b]$.

• We can use the IVT to show that a function has a certain value within a given interval. More specifically, if a continuous function is positive at one point and negative at another, then it must be zero somewhere in between.

The Extreme Value Theorem: If a function f is continuous on a closed interval $[a, b]$, then f takes on a minimum and maximum value at least once in $[a, b]$.

• *Intuitively*, the idea is that a continuous function in a closed interval must have extrema. Since continuous functions have no jumps or gaps, there must be a highest value and a least value of the function on any closed interval.

• In practice, we know that these extrema must occur either at a critical point or an end point (that is, at the top of a "hill", at the bottom of a "valley", or on the way to a more extreme value when the boundary of the interval "gets in the way"). Therefore, we find the extrema of a continuous function on closed interval by:

1. Finding all critical numbers
2. Computing the value of the function on all critical numbers **inside** the interval in question
3. Computing the value of the function at the boundary points of the interval
4. Compare all these values: the biggest value is the max, the smallest in the min.

Rolle's Theorem: If a function f is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$, then $f'(c) = 0$ for at least one number c in (a, b).

• *Intuitively,* this theorem says that if a differentiable function attains the same value twice, then it must have a "turning point" somewhere in between.

The Mean Value Theorem: If a function f is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), then there exists a number c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. Or, in other words, where $f(b) - f(a) = f'(c)(b - a)$

• *Intuitively,* this theorem says that if we pick any pair of points on a differentiable function and compute the slope of the secant line between those two points, then there is a point c in between whose tangent line slope is the same as the slope of the previous secant line.

Example:

Let $f(x) = x^3 - 3x$. Notice that f is continuous since it is a polynomial, and since $f'(x) = 3x^2 - 3$ is also a polynomial, then f is also differentiable.

1. To see how the IVT applies to this function, notice that $f(-1) = -1 + 3 = 2$ while $f(1) = 1 - 3 = -2$,so there must be a c between -1 and 2 such that $f(c) = 0$. Of course it is easy to see where the root is, since we can easily notice and verify that $f(0) = 0$.

2. To see how the EVT can be applied to this function, let's consider f on the interval [0, 3]. Since $f'(x) = 3x^2 - 3$, the critical numbers of $f(x)$ occur when $3x^2 - 3 = 0$, or $3x^2 = 3$. That is, when $x^2 = 1$, or $x = \pm 1$. However, notice that $x = -1$ is outside the interval we are considering, so to find the extrema of f on [0,3], we only check the values of f when $x = 0, 1, 3$.

   Notice that $f(0) = 0$, $f(1) = -2$, and $f(3) = 27 - 9 = 18$. Therefore, on this interval, the maximum value of f is 18, and the minimum value of f is -2.

3. To see one way that Rolle's Theorem can be applied to this function, notice that $f(x) = 0$ when $x^3 - 3x = 0$, or when $x(x^2 - 3) = 0$. That is, when $x = 0$, and when $x = \pm\sqrt{3}$. Therefore, according to Rolle's Theorem, there must be an x-value between $-\sqrt{3}$ and zero where $f'(x) = 0$, and there must also be an x-value between zero and $\sqrt{3}$ where $f'(x) = 0$. [This does in fact end up being true, since we have already seen that $f'(-1) = 0$, and $f'(1) = 0$].

4. To see how the Mean Value Theorem applies to this function, consider the function f on the interval [0,2]. Notice that $f(0) = 0$, and $f(2) = 2^3 - 3(2) = 8 - 6 = 2$, so the endpoints of this interval are $(0, 0)$ and $(2, 2)$. Therefore, the slope of the secant line between these endpoints is: $m_{sec} = \frac{2-0}{2-0} = \frac{2}{2} = 1$. The MVT claims that there should be at least one x-value c between 0 and 2 for which $f'(c) = 1$. Let's find it:

   Suppose $f'(c) = 1$. Then $3c^2 - 3 = 1$, so $3c^2 = 4$. Therefore, $c^2 = \frac{4}{3}$, so $c = \pm\sqrt{\frac{4}{3}} = \pm\frac{2}{\sqrt{3}} = \pm\frac{2\sqrt{3}}{3}$. Notice that $\frac{2\sqrt{3}}{3} \approx 1.1547$, so we have found a c value in the predicted interval.

Note: On our next lab, we will also look at how to Apply the these theorems when we only have a table of values for our function and its derivative rather than actual equations for our functions.