Math 127
Quadratic, Polynomial and Rational Inequalities Handout

Example 1: Consider the inequality $x^2 - x - 6 > 0$. If we jump in and use algebra to solve this inequality, we notice that we can factor the left hand side to get : $(x-3)(x+2) > 0$. If this were an equation, we would use the zero factor property to "split" the equation into two linear factors since a product is zero only if at least one of its factors is zero. However, here, we need the product of these terms to be **positive**. For this to happen, there are two possibilities:

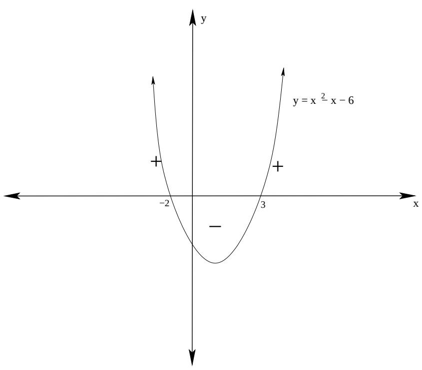
1. Both factors could be positive in which case, the product of two positive numbers is positive.

2. Both factors could be negative in which case, the product of two negative numbers is positive.

Therefore, to solve this inequality, we need a way of determining the sign of the product of the terms $(x-3)$ and $(x+2)$ for various values of x. We will look at two ways of doing this.

Method 1: The Graphical Method

Notice that instead of thinking of this inequality, we can instead look at the related function $f(x) = x^2 - x - 6 = (x-3)(x+2)$. This is a quadratic function with zeros at $x = 3$ and at $x = -2$. Since the leading coefficient is positive, we know that the graph of this function is a parabola that open upward. We aren't really interested in the location of the vertex (although we could find it if we wanted to), so we will just make a rough graph of $f(x)$ using the information we already have:

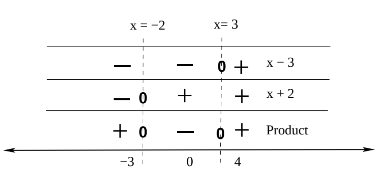


Notice that we can see from our graph that $f(x) = 0$ when $x = -2$ and $x = 3$, $f(x) < 0$ for $-2 < x < 3$ and that $f(x) > 0$ for $x < -2$ and for $x > 3$.
Therefore, the solution to this inequality (in interval notation) is: $(-\infty, -2) \cup (3, \infty)$.

Method 2: The Sign Chart Method

Sometimes, we do not want to take the time to carefully graph a polynomial related to an inequality (especially if it has degree ≥ 3). In these cases, an alternative is to use a **sign chart** to find the sign of the result of multiplying the factors of the polynomial on each interval of possible inputs. We then use the results of this sign chart to solve the inequality. We will redo the previous example using the sign chart method.

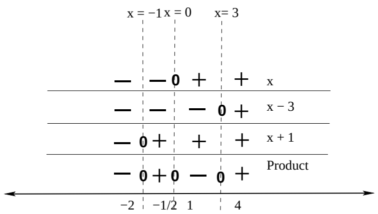
As above, the factored form of this inequality is $(x-3)(x+2) > 0$. The *Key Values* for our sign chart are the zeros of the related polynomial − in this case, $x = -2$ and $x = 3$. These are the inputs that yield zero, so any sign changes must occur at these values.
Our chart will have a row for each factor. We then place a sign in each box within the chart based on whether the factor in that row is positive or negative for the related inputs. We can use a "test value" or our understanding of the factor to choose the appropriate sign.
In order to fill in the final row in our chart, we "multiply" the signs in each column in order to compute the "total sign" for that interval of inputs. We will usually also take the time to fill in the values where the resulting sign is zero (or undefined in the case of a rational inequality).



Notice that, as before, the final row in our sign chart shows that the solution to this inequality (in interval notation) is: $(-\infty, -2) \cup (3, \infty)$.

Example 2: Now consider the inequality $x^3 - 2x^2 - 3x \leq 0$.

As before, we can solve this using either the Graphical Method or the Sign Chart Method. We will use the sign chart method. Notice that the factored form of this polynomial inequality is $x(x^2 - 2x - 3) \leq 0$, or $x(x-3)(x+1) \leq 0$. Then the Key Values for this polynomial inequality are: $x = 0$, $x = 3$, and $x = -1$. We build the sign chart, making a separate row for each factor, followed by a final row giving the sign of the product for each sub-interval.



Since our inequality is not strict, we will include the endpoints of the intervals in our final answer (values where $f(x) = 0$ are included in the solution set). Therefore, the solution is: $(-\infty, 1] \cup [0, 3]$.

Example 3: Consider the inequality $\dfrac{x^2 + 4x - 5}{x - 2} \geq 0$.

We will solve this inequality using a sign chart, but there is one small complication here. Notice that the this inequality is a rational expression rather than a polynomial. Hence this expression is undefined when $x = 2$. This will still be a Key Value for our sign chart and it will be the dividing line indicating the sign change due to the term $(x - 2)$. However, in our final answer, we must realize that the related function is undefined on this input (rather than zero), so i must be excluded from out solution set.

Example 4: Consider the inequality $\dfrac{x - 1}{x + 1} \leq 4$.

We have still one more complication here. Notice that in this inequality, the terms are not being compared to zero. This is a big deal, since our entire method is based on finding the sign of the expression on each sub-interval. For our method, we **must** use algebra to rewrite this inequality with zero on one side and an expression in simplified factored form on the other side. We begin by simplifying algebraically, then we will make our sign chart.