3.2.3 Affine Transformation of the Euclidean Plane

A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.
— Godfrey Harold Hardy (1877–1947)

What is the form of a transformation matrix for the analytic model of the Euclidean plane? We investigate this question. Let $A = [a_{ij}]$ be a transformation matrix for the Euclidean plane and $(x, y, 1)$ be any point in the Euclidean plane. Then

$$
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  1
\end{bmatrix} =
\begin{bmatrix}
  a_{11}x + a_{12}y + a_{13} \\
  a_{21}x + a_{22}y + a_{23} \\
  a_{31}x + a_{32}y + a_{33}
\end{bmatrix}
$$

Since the last matrix must be the matrix of a point in the Euclidean plane, we must have $a_{31}x + a_{32}y + a_{33} = 1$ for every point $(x, y, 1)$ in the Euclidean plane. In particular, the point $(0, 0, 1)$ must satisfy the equation. Hence, $a_{33} = 1$. Further, the points $(0, 1, 1)$ and $(1, 0, 1)$ satisfy the equation and imply $a_{32} = 0$ and $a_{31} = 0$, respectively. Therefore, the transformation matrix must have the form

$$
A = 
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  0 & 0 & 1
\end{bmatrix},
$$

which motivates the following definition.

Definition. An affine transformation of the Euclidean plane, $T$, is a mapping that maps each point $X$ of the Euclidean plane to a point $T(X)$ of the Euclidean plane defined by $T(X) = AX$ where $\det(A)$ is nonzero and

$$
A = 
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  0 & 0 & 1
\end{bmatrix},
$$

where each $a_{ij}$ is a real number.

Exercise 3.19. Prove that every affine transformation of the Euclidean plane has an inverse that is an affine transformation of the Euclidean plane. (Hint. Write the inverse by using the adjoint. Refer to a linear algebra text.)

Proposition 3.3. An affine transformation of the Euclidean plane is a transformation of the Euclidean plane.

Exercise 3.20. Prove Proposition 3.3.

Click here to see an animation of a sequence of affine transformations.
Proposition 3.4. The set of affine transformations of the Euclidean plane form a group under matrix multiplication.

Proof. Since the identity matrix is clearly a matrix of an affine transformation of the Euclidean plane and the product of matrices is associative, we need only show closure and that every transformation has an inverse.

Let $A$ and $B$ be the matrices of affine transformations of the Euclidean plane. Since $\det(A)$ and $\det(B)$ are both nonzero, we have that $\det(AB) = \det(A) \cdot \det(B)$ is not zero. Also,

$$AB = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
b_{11} & b_{12} & b_{13} \\
b_{21} & b_{22} & b_{23} \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} + a_{13} \\
a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} + a_{23} \\
0 & 0 & 1
\end{bmatrix}$$

is a matrix of an affine transformation of the Euclidean plane. *(The last row of the matrix is 0, 0, 1.)* Hence closure holds.

Complete the proof by showing the inverse property.//

Exercise 3.21. Given three points $P(0, 0, 1)$, $Q(1, 0, 1)$, and $R(2, 1, 1)$, and an affine transformation $T$.
(a) Find the points $P' = T(P)$, $Q' = T(Q)$, and $R' = T(R)$ where the matrix of the transformation is $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$. (b) Sketch triangle $PQR$ and triangle $P'Q'R'$. (c) Describe how the transformation moved and changed the triangle $PQR$.

Exercise 3.22. Find the matrix of an affine transformation that maps $P(0, 0, 1)$ to $P'(0, 2, 1)$, $Q(1, 0, 1)$ to $Q'(2, 1, 1)$, and $R(2, 3, 1)$ to $R'(7, 9, 1)$.

Exercise 3.23. Show the group of affine transformations of the Euclidean plane is not commutative.