Math 310, Discrete Mathematics
Semester: Spring 2015
Instructor: Damiano Fulghesu

Keep in mind that the following exercises only cover the technique part of the study guidelines. On the test you will also be asked some questions about the knowledge part of the study guidelines.

1. Let $U := \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{2, 5\}$, $B = \{5, 6, 8\}$, and $C = \{1, 2, 4, 7, 9, 10\}$. Determine the following sets/cardinalities.
   
   (a) $B \setminus C$
   (b) $A \cup (B \cap C)$
   (c) $A \times B$
   (d) $A \oplus C$
   (e) $|\mathcal{P}(C)|$
   (f) $|\mathcal{P}(\mathcal{P}(A))|$
   (g) $|B \times C|$
   (h) $|A \times B \times C|$

2. Let $A = \{\emptyset, \{1, 2\}\}$. True or False.
   
   (a) $\emptyset \subseteq A$
   (b) $\emptyset \in A$
   (c) $\{\emptyset\} \subseteq A$
   (d) $\{\emptyset\} \in A$
   (e) $2 \in A$
   (f) $\{2\} \subseteq A$
   (g) $\{1, 2\} \in A$
   (h) $\{1, 2\} \subseteq A$
   (i) $\{\{1, 2\}\} \subseteq A$

3. Use Venn diagrams to show whether or not the following identity is true
   
   $$A \cap (B \setminus C) = (A \cap B) \oplus (A \cap C)$$

4. Prove the following equality
   
   $$(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$$

   by proving that each side is a subset of the other side.

5. Let $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(n) = n^2 - 17$. Is $f$ one-to-one? Is $f$ onto? Justify your answers.

6. Let $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n^2 - 17$. Is $f$ one-to-one? Is $f$ onto? Justify your answers.

7. Let $f : \{1, 2, 3, 4\} \to \{-1, -8, -13, -16\}$ defined by $f(n) = n^2 - 17$. Is $f$ one-to-one? Is $f$ onto? Justify your answers.

8. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $f(a, b) = 3a + 2b$. Determine:
   
   (a) $f(n, n)$
   (b) $f(2, -3)$
   (c) $f^{-1}(\{0\})$
   (d) $f^{-1}(E)$ where $E$ is the set of even integers.
   (e) Determine the range of $f$. Justify your answer.
9. Let $R$ be the relation on $\mathbb{N}$ defined by:

$(a, b) \in \mathbb{N}$ iff $a$ divides $b$.

(a) Say whether $R$ is reflexive.
(b) Say whether $R$ is symmetric.
(c) Say whether $R$ is antisymmetric.
(d) Say whether $R$ is transitive.
(e) Is there an element $b \in \mathbb{N}$ such that $(a, b) \in \mathbb{N}$ for all $a \in \mathbb{N}$?
(f) Is there an element $a \in \mathbb{N}$ such that $(a, b) \in \mathbb{N}$ for all $b \in \mathbb{N}$?

Justify your answers.

10. Prove by induction that for all integers $n \geq 1$ we have

$$\sum_{i=1}^{n} i^3 = \left[ \frac{n(n+1)}{2} \right]^2.$$

11. Prove that 5 divides $n^5 - n$ whenever $n$ is a nonnegative integer.

12. Prove that $n! < n^n$ for all integers $n > 1$.

13. Let $f(x) = xe^x$. Prove that $g^{(n)}(x) = (x + n)e^x$ for all integers $n \geq 1$.

14. Assume that a chocolate bar consists of $n$ squares arranged in a rectangular pattern. The entire bar, or any smaller rectangular piece of the bar, can be broken along a vertical or a horizontal line separating the squares. Assuming that only one piece can be broken at a time, determine how many breaks you must successively make to break the bar into $n$ separate squares. Use strong induction to prove your answer.

15. Consider the following relation on $\{1, 2, 3\}$.

$$R := \{(1, 1), (1, 2), (1, 3), (2, 3)\}$$

Represent $R$ with a matrix and with a directed graph.

16. Draw any directed graph corresponding to a symmetric and reflexive relations of a set with 5 elements.

17. Let $R$ be the relation on $\mathbb{Z}$ defined by $R := \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m + n$ is even $\}$. Determine whether or not $R$ is an equivalence relation. If $R$ is an equivalence relation, describe its equivalence classes.