Math 262
Exam 1 Solutions

Instructions: You will have 50 minutes to complete this exam. Calculators are allowed, but this is a closed book, closed notes exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit. Simplify answers when possible and follow directions carefully on each problem.

1. (12 points) Find the area of the region bounded by the graph \( y = 3x - x^2 \) and the \( x \)-axis.

Notice that if \( 3x - x^2 = 0 \), then \( x(3 - x) = 0 \), so either \( x = 0 \) or \( x = 3 \). In fact, the graph of the region enclosed by \( y = 3x - x^2 \) and the \( x \)-axis is as follows:

\[
\begin{align*}
\text{Therefore, } A &= \int_{0}^{3} 3x - x^2 \, dx = \left[ \frac{3}{2} x^2 - \frac{1}{3} x^3 \right]_0^3 \\
&= \frac{3}{2}(9) - \frac{1}{3}(27) - 0 = \frac{27}{2} - \frac{9}{2} = \frac{27}{2} - \frac{18}{2} = \frac{9}{2} \text{ units}^2
\end{align*}
\]

2. (12 points) Find the centroid of the region bounded by the graph \( y = 3x - x^2 \) and the \( x \)-axis, given that the lamina is made from material with a uniform density of 2 lbs per each unit of area. [Hint: one coordinate can be found quickly by using symmetry].

First notice that \( m = \rho \cdot A = \frac{2 \text{ lbs}}{\text{units}^2} \cdot \frac{9}{2} \text{ units}^2 = 9 \) lbs.

Next, by symmetry, \( \overline{x} = \frac{3}{2} \).

Recall \( M_x = \int_{-a}^{b} \frac{1}{2} [f(x) + g(x)] \rho[f(x) - g(x)] \, dx = \int_{0}^{3} \frac{1}{2} [(3x - x^2) + (0)][(3x - x^2) - (0)] \, dx = \int_{0}^{3} (3x - x^2)^2 \, dx \)

\[
= \int_{0}^{3} 9x^2 - 6x^3 + x^4 \, dx = 3x^3 - 3 \cdot \frac{x^4}{2} + \frac{1}{5} x^5 \bigg|_0^3 = 81 - 3 \cdot \frac{27}{2} + \frac{1}{5} (243) = \frac{810}{10} - \frac{1215}{10} + \frac{486}{10} = \frac{81}{10}
\]

Thus \( \overline{y} = \frac{M_x}{m} = \frac{\frac{81}{10}}{9} = \frac{81}{90} = \frac{9}{10} \), and hence \( (\overline{x}, \overline{y}) = \left( \frac{3}{2}, \frac{9}{10} \right) \)
3. (12 points) The base of a certain solid is a region in the \(xy\)-plane is bounded by the curves \(y = 2x - 1\) and \(y = x^2 - 3x + 3\). Set up a definite integral that represents the volume of this solid if every cross section by a plane perpendicular to the \(x\)-axis is a square. You DO NOT need to evaluate this integral.

Notice that if \(2x - 1 = x^2 - 3x + 3\), then \(0 = x^2 - 5x + 4\), or \(0 = (x - 4)(x - 1)\), so either \(x = 4\) or \(x = 1\). In fact, the graph of the region enclosed by \(y = 2x - 1\) and \(y = x^2 - 3x + 3\) is as follows:

Since the cross sections of this solid are squares that lie in planes perpendicular to the \(x\)-axis, then the side length of the cross sections are given by: \(s(x) = (2x - 1) - (x^2 - 3x + 3) = -x^2 + 5x - 4\)

Therefore the volume of this solid is given by: \(V = \int_{1}^{4} (-x^2 + 5x - 4)^2 \, dx\).

4. (12 points) A freight elevator weighing 3000 pounds is supported by a 50-foot-long cable that weighs 3 pounds per linear foot. Find the work required to lift the elevator 10 feet by winding the cable onto a winch.

The work done on the elevator is: \(W_e = (3000 \text{ lbs.})(10 \text{ ft.}) = 30,000 \text{ ft.-lbs.}\)

Notice that initially there are 50 feet of cable with a density of \(3 \text{ lbs/ft.}\).

After the elevator is lifted up 10 feet, there are only 40 feet of cable still unwound. Therefore, the work done on the cable is given by:

\[
W_c = \int_{0}^{10} (3 \text{ lbs./ft.})(50 - x) \, dx = \int_{0}^{10} 150 - 3x \, dx = 150x - \frac{3}{2}x^2 \bigg|_{0}^{10} = 1,500 - 150 = 1350 \text{ ft.-lbs.}
\]

This \(W = W_e + W_c = 30,000 + 1350 = 31,350 \text{ ft.-lbs.}\)
5. (8 points each) Set up a definite integral representing the volume of the region formed when the region bounded by \( y = 2x - 1 \) and \( y = x^2 - 3x + 3 \) is revolved about:

(a) the \( x \)-axis

![Diagram showing revolution about the x-axis](image)

Notice that it is most convenient to slice vertically and to use washers. In this case, \( r_o = 2x - 1 \) and \( r_i = x^2 - 3x + 3 \)

Then \( V = \int_1^4 \pi \left[ (2x - 1)^2 - (x^2 - 3x + 3)^2 \right] \, dx \)

(b) the \( y \)-axis

![Diagram showing revolution about the y-axis](image)

Notice that it is most convenient to slice vertically and to use cylindrical shells. In this case, \( r = x \) and \( h = (2x - 1) - (x^2 - 3x + 3) \)

Then \( V = \int_1^4 2\pi x \left[ (2x - 1) - (x^2 - 3x + 3) \right] \, dx \)
(c) the line $x = 7$

\begin{equation*}
\text{Notice that it is most convenient to slice vertically and to use cylindrical shells. In this case, } r = 7 - x \text{ and } h = (2x - 1) - (x^2 - 3x + 3).
\end{equation*}

Then $V = \int_{1}^{4} 2\pi(7 - x) \left[ (2x - 1) - (x^2 - 3x + 3) \right] \, dx$

(d) the line $y = 12$

\begin{equation*}
\text{Notice that it is most convenient to slice vertically and to use washers. In this case, } r_o = 12 - (x^2 - 3x + 3) \text{ and } r_i = 12 - (2x - 1).
\end{equation*}

Then $V = \int_{1}^{4} \pi \left[ (12 - (x^2 - 3x + 3))^2 - (12 - (2x - 1))^2 \right] \, dx$

6. (10 points) Set up a definite integral representing the arc length of a wire in the shape of the function $f(x) = 2\sqrt{\sin x}$ for $0 \leq x \leq \pi$.

First notice that $f'(x) = \frac{1}{2} (2)(\sin x) \frac{1}{2} \cos x = \frac{\cos x}{\sqrt{\sin x}}$

Recall that $L = \int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx$

Thus $L = \int_{0}^{\pi} \sqrt{1 + \frac{\cos^2 x}{\sin x}} \, dx$
7. (15 points) A water trough has a rectangular top that is 18 feet long and 6 feet wide. Each vertical cross section of the trough is an equilateral triangle with side length 6 feet. Set up a definite integral representing the amount work required to fill the tank with water, given that water weighs 62.4 lb/feet$^3$. Assume that the bottom of the tank is at ground level. You DO NOT need to evaluate this integral.

![Diagram of a water trough with dimensions labeled](image)

Notice that it is most convenient to slice horizontally, since we want to slice in such a way that each cross section can be thought of as some force being moved through a specific distance $d$. Notice that the cross sections are rectangles of length 18 ft and whose width varies as a function of the depth of the slice.

In order to find the volume of each cross section as a function of depth, we need to find a way to express the width of a slice in terms of its depth. Notice that, using the Pythagorean Theorem along with the fact that vertical slices are equilateral triangles with side length 6 feet, the depth of the tank is given as follows:

\[ h^2 = 6^2 - 3^2 = 36 - 9 = 27, \text{ so } h = \sqrt{27} = 3\sqrt{3} \text{ ft.} \]

Moreover, using trigonometry, the width of the slice at height $y$ is given by $w(y) = \frac{2\sqrt{3}y}{3}$.

Finally, $A = l \cdot w$, so $A(y) = (18) \frac{2\sqrt{3}}{3}y = 12\sqrt{3}y$ ft$^2$, so we have:

\[ W = \int_a^b A(y)\rho \, dy = \int_0^{3\sqrt{3}} (12\sqrt{3}y)(62.4) \, dy \]