Recall: A set of vector $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ in a vector space $V$ forms a basis for $V$ if both of the following hold:

(a) $\text{span } S = V$ (that is, the vector space $V$ is spanned by the set $S$)  
(b) $S$ is a linearly independent set.

Examples:

1. 

2. 

3. 

Definition: A vector space $V$ is **finite dimensional** if there is a finite subset of $V$ that is a basis for $V$. If no such subspace exists, then we say that $V$ is **infinite dimensional**.

Note: If $S_1 = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a basis for $V$, then $S_2 = \{c\vec{v}_1, c\vec{v}_2, \ldots, c\vec{v}_k\}$ for $c \neq 0$ is also a basis, so bases aren’t unique.

Theorem 4.8: If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a basis for a vector space $V$ then every vector in $V$ can be written in one and only one way as a linear combination of vectors in $V$.

Proof: Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a basis for a vector space $V$ and suppose $\vec{v} \in V$. Since $S$ spans $V$, there is at least one way of expressing $\vec{v}$ as a linear combination in $S$. Suppose that $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k$ and $\vec{v}_1 = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_k\vec{v}_k$. Then $0 = \vec{v} - \vec{v} = (a_1 - b_1)\vec{v}_1 + (a_2 - b_2)\vec{v}_2 + \cdots + (a_k - b_k)\vec{v}_k$. Since $S$ is linearly independent, we must have $(a_1 - b_1) = (a_2 - b_2) = \cdots = (a_k - b_k) = 0$. That is, $a_i = b_i$ for all $i = 1 \cdots k$. □

Theorem 4.9: Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of non-zero vectors in a vector space $V$ and suppose $W = \text{span } S$. Then some subset of $S$ is a basis for $W$.

Proof: First, notice that if $S$ is linearly independent, then $S$ is a basis for $W$.

Suppose that $S$ is linearly dependent. Then, using Theorem 4.7, some vector $v_j$ in $S$ can be written as a linear combination of the other vectors in $S$. Hence, we may delete this vector from $S$ to obtain a strictly smaller set $S_1$ that still spans $W$. If $S_1$ is linearly independent, then it is a basis for $W$. Otherwise, we may apply the same procedure to delete a vector from $S_1$. Since a single non-zero vectors is linearly independent and $S$ is a finite set, repeating this procedure must eventually produce a basis for $W$.

Note: The proof of Theorem 4.9 suggests the following algorithm for finding a basis of a subspace $W$ of a vector space $V$:

- **Step 0:** Begin with a set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ such that $\text{span } S = W$.

- **Step 1:** Consider the equation $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = \vec{0}$. Solve this system for $a_1, a_2, \ldots, a_k$ by representing this system as a matrix and applying the Gauss-Jordan method to put the augmented matrix into reduced row echelon form. If $a_1 = a_2 = \cdots = a_k = 0$, then $S$ is already a basis.

- **Step 2:** If not, find a vector $v_j$ that is a linear combination of the other vectors in $S$ and delete it from the set $S$, obtaining a smaller set $S_1$.

- Repeat Steps 1 and 2 for the new set $S_1$. This process will eventually produce a basis for $W$. 


Special Case: If \( V = \mathbb{R}^n \) or \( V = R_n \), then the following more efficient method can be used:

- **Step 0:** Begin with a set \( S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\} \) such that \( \text{span} \ S = W \).

- **Step 1:** Consider the equation \( a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = \vec{0} \). Solve this system for \( a_1, a_2, \cdots, a_k \) by representing this system as a matrix and applying the Gauss-Jordan method to put the augmented matrix into reduced row echelon form.

- **Step 2:** The collection of vectors corresponding to the columns that contain a leading 1 form a basis for \( W \).

**Theorem 4.10:** If \( S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\} \) is a basis for a vector space \( V \), and \( T = \{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_r\} \) is a linearly independent set in \( V \), then \( r \leq n \)

**Proof:**

**Corollary 4.1:** If \( S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\} \) and \( T = \{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_m\} \) are bases for a vector space \( V \), then \( m = n \).

**Proof:** Applying Theorem 4.10 to \( S \) and \( T \), we have that \( m \leq n \). Reversing the roles of the two sets and applying Theorem 4.10 again, we have \( n \leq m \). Hence \( m = n \). \( \square \)

**Notes:** A single vector space \( V \) can have many different bases.

**Definition:** The **dimension** of a non-zero vector space \( V \) is the number of vectors in a basis for \( V \). This is well defined by Corollary 4.1. We denote this as: \( \text{dim} \ V \).

**Note:** By convention, \( \text{dim} \ \{\vec{0}\} = 0 \).

**Examples:**

- Both \( \mathbb{R}^n \) and \( R_n \) have dimension \( n \). (What is the dimension of \( M_{mn} \)?)
- \( P_n \) has dimension \( n + 1 \) (for example, since \( P_2 = \{p(t) : p(t) = at^2 + bt + c\} \), then \( \text{dim} \ P_2 = 3 \)).
- \( P \) is infinite dimensional.

**Definition:** Let \( S \) be a set of vectors in a vector space \( V \). A subset \( T \) of \( S \) is called a **maximal independent subset** of \( S \) if \( T \) is a linearly independent set of vectors that is not properly contained in any other linearly independent subset of \( S \). Similarly, a **minimal spanning set** of a vector space \( V \) is a set \( S \) of vectors that spans \( V \) and that does not contain any proper subset that spans \( V \).

**Corollary 4.2:** If \( \text{dim} \ V = n \), then any maximal independent subset of \( V \) contains \( n \) vectors.

**Corollary 4.3:** If a vector space \( V \) has dimension \( n \), then any minimal spanning set of \( V \) contains \( n \) vectors.

**Corollary 4.4:** If a vector space \( V \) has dimension \( n \), then any set of \( m > n \) vectors is linearly dependent.

**Corollary 4.5:** If a vector space \( V \) has dimension \( b \), then any set of \( m < n \) vectors does not span \( V \).