This miniproject asks you to find and explain the error in reasoning in three proofs. For two of them you will need the following information.

A *Pythagorean triple* is an ordered triple of integers (a, b, c) such that $a^2 + b^2 = c^2$. Note that if (a, b, c) is a Pythagorean triple, so is (b, a, c). Similarly, a Pythagorean quadruple is an ordered four-tuple (a, b, c, d) such that $a^2 + b^2 + c^2 = d^2$. As an example, note that

$(3, 4, 5)$ is a Pythagorean triple since $3^2 + 4^2 = 5^2$,

$(5, 12, 13)$ is a Pythagorean triple since $5^2 + 12^2 = 13^2$,

and $(3, 4, 12, 13)$ is a Pythagorean quadruple since $13^2 = 5^2 + 12^2 = \left(3^2 + 4^2\right) + 12^2 = 3^2 + 4^2 + 12^2$.

(a) Explain why the argument given in #34 of Section 1.6 of the textbook does not work.

(b) Consider the following theorem and the two proofs given.

Theorem For every Pythagorean quadruple (a, b, c, d) there exists some order of the arguments a, b, c and an integer e such that (a, b, e) and (e, c, d) are both Pythagorean triples.

Proof 1 Since (a, b, c, d) is a Pythagorean quadruple, we know that

$$a^2 + b^2 + c^2 = d^2.$$

But, by subtracting c^2 for both sides, this tells us that

$$a^2 + b^2 = d^2 - c^2.$$

Therefore, let $e^2 = d^2 - c^2$. Then we have both of the following

$$a^2 + b^2 = e^2 \text{ (by definition of } e\text{), and}$$

$$e^2 + c^2 = \left(d^2 - c^2\right) + c^2 = d^2,$$

which proves the theorem. □

Proof 2 Since (a, b, c, d) is a Pythagorean quadruple, we know that

$$a^2 + b^2 + c^2 = d^2.$$

Assume that c is the largest of the three integers, so $c \geq a$ and $c \geq b$, so that $a^2 + b^2 = c^2$. Then for $e = c$ we have

$$a^2 + b^2 = c^2 = e^2$$

$$\text{and } e^2 + c^2 = a^2 + b^2 + c^2 = d^2,$$

which is what we wished to prove. □

Note that *neither* proof is correct. Do each of the following.

(i) Find and explain the flaw of reasoning in the first proof.

(ii) Find and explain the flaw of reasoning in the second proof.

(iii) Is the theorem really a theorem? That is, is it correct? Explain.