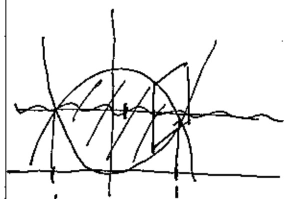
68

$56-62 \Rightarrow 9$
$49-55 \Rightarrow 8$
$43-48 \Rightarrow 7$
$36-42 \Rightarrow 6$
$29-35 \Rightarrow 5$

Math 262 Calculus II Lab 5 Volumes by Cross Sections Name: KEY

1. The base of a solid is the region bounded by $y = x^2$ and $y = 2 - x^2$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is: $\int_{-1}^{1} s^2 ds$ (+1)

(a) a square



$x^2 = 2 - x^2$
$2x^2 = 2$
$x^2 = 1$
$x = \pm 1$ (+1)
$y = 1$

$V = \int_{-1}^{1} (2-x^2-x^2)^2 dx$ 5 (+2)

$= \int_{-1}^{1} (2-2x^2)^2 dx$

$= 4 \int_{-1}^{1} (1-x^2)^2 dx$

$= 4 \left[\int_{-1}^{1} (1-2x^2+x^4) dx \right]$ (+1)

d (+1) $= 4 \left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \right]_{-1}^{1}$ (+1)

$= 4 \left[1 - \frac{2}{3} + \frac{1}{5} + 1 - \frac{2}{3} + \frac{1}{5} \right]$ (+2)

$= \frac{64}{15}$

8 pts

(b) a semicircle $\int_{-1}^{1} \frac{1}{2}\pi r^2 dx$ (+1)

$V = \frac{1}{2}\pi \int_{-1}^{1} \left(\frac{2-x^2-x^2}{2} \right)^2 dx$ r (+2)

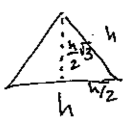
(+1)

$= \frac{\pi}{2} \int_{-1}^{1} (1-x^2)^2 dx$

$= \frac{\pi}{2} \left[\frac{16}{15} \right] = \frac{8\pi}{15}$ (+3)

8

(c) an equilateral triangle



$A = \frac{1}{2}\left(\frac{h\sqrt{3}}{2}\right)(h) = \frac{\sqrt{3}}{4} h^2$ (+1)

$V = \int_{-1}^{1} \frac{\sqrt{3}}{4} (2-x^2-x^2)^2 dx$ h (+2)

(+1)

$= \frac{\sqrt{3}}{4} 4 \int_{-1}^{1} (1-x^2)^2 dx$

$= \sqrt{3} \left(\frac{16}{15} \right) = \frac{16\sqrt{3}}{15}$ (+4)

8

Page 1 Show all Work for Credit Give exact answers unless otherwise noted

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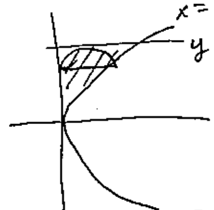
Math 262 Calculus II Lab 5 Volumes by Cross Sections Name: KEY

2. The base of a solid is the region bounded by $x = y^2\sqrt{5}$, $y = 2$, and the y-axis.

(a) Find the volume of the solid if every cross section by a plane perpendicular to the y-axis is a semicircle.



$x = y^2\sqrt{5}$
$y = 2$

Sketch (+1)

$V = \int_0^2 \frac{1}{2}\pi r^2 dy$ (+1)

(+1)

$= \frac{\pi}{2} \int_0^2 \left(\frac{y^2\sqrt{5}}{2} \right)^2 dy$ r (+2) (d, r)

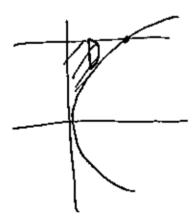
$= \frac{5\pi}{8} \int_0^2 y^4 dy$ (+1)

$= \frac{\pi}{8} y^5 \Big|_0^2$ (+1)

$= 4\pi$ (+1)

8 pts

(b) Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is a semicircle.



$x = y^2\sqrt{5}$
$y = \left(\frac{x}{\sqrt{5}}\right)^{1/2}$

$x = 4\sqrt{5}$

$V = \int_0^{4\sqrt{5}} \frac{1}{2}\pi r^2 dx$ (+1)

y = (+1)
d (+1)
r (+1)

$= \frac{\pi}{2} \int_0^{4\sqrt{5}} \left(\frac{2 - \frac{\sqrt{x}}{\sqrt[4]{5}}}{2} \right)^2 dx$

(+1.5) $= \frac{\pi}{8} \int_0^{4\sqrt{5}} \left(4 - \frac{4}{\sqrt[4]{5}}\sqrt{x} + \frac{x}{\sqrt{5}} \right) dx$

(+1.5) $= \frac{\pi}{8} \left[4x - \frac{4}{\sqrt[4]{5}} \cdot \frac{2}{3} x\sqrt{x} + \frac{1}{2\sqrt{5}} x^2 \right]_0^{4\sqrt{5}}$

$= \frac{\pi}{8} \left[16\sqrt{5} - \frac{8}{3\sqrt[4]{5}} 4\sqrt{5}\, 2\sqrt[4]{5} + \frac{1}{2\sqrt{5}} 16 \cdot 5 \right]$

(+2) $= \frac{\pi}{8} \left[16\sqrt{5} - \frac{64}{3}\sqrt{5} + 8\sqrt{5} \right]$

$= \frac{\pi}{8} \left(\frac{8\sqrt{5}}{3} \right) = \frac{\pi\sqrt{5}}{3}$

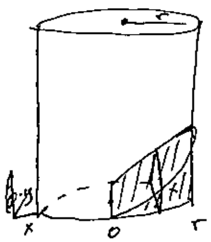
10

Page 2 Show all Work for Credit Give exact answers unless otherwise noted

$\frac{18}{42}$

(23)

3. **Volume of a Wedge**
A plane inclined at an angle of 45° passes through a diameter of the base of a right circular cylinder of radius r. Find the volume of the region within the cylinder and below the plane. (This volume was first calculated in the third century BCE by Archimedes, who also derived the formula $V = \frac{4}{3}\pi r^3$ for the volume of a sphere of radius r. His work on the wedge is found in a manuscript that was discovered in 1906 after having been lost for centuries.)



$V = \int_0^r lw\,dx$ (+1) (+1)

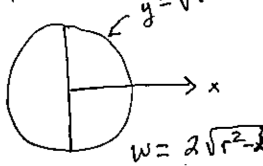
$= \int_0^r 2x\sqrt{r^2-x^2}\,dx$ (+1) $\quad u = r^2-x^2$, $du = -2x\,dx$

$= \frac{-2}{2}\int_{r^2}^0 \sqrt{u}\,du = -\frac{2}{3}u^{3/2}\Big|_{r^2}^0$ (+3)

$= -0 + \frac{2}{3}(r^2)^{3/2}$

$= \frac{2}{3}r^3$ (+2)

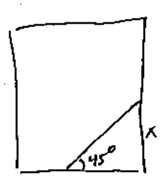
Base



$y = \sqrt{r^2-x^2}$

$w = 2\sqrt{r^2-x^2}$ (+2)

Center



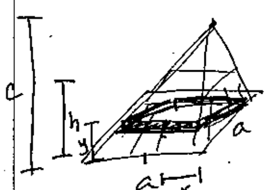
$l = x$ (+2)

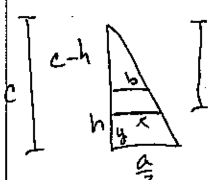
12

$\frac{12}{54}$

4. A frustum of a right pyramid is a right pyramid with its top cut off. Let V be the volume of a frustum of height h whose base is a square of side a and top is a square of side b with $a > b \geq 0$. Show that $V = \frac{h}{3}(a^2 + ab + b^2)$. (A papyrus dating to the year 1850 BCE indicates that Egyptian mathematicians had discovered this formula almost 4,000 years ago.) [Hint: This problem can be done by modifying the method of example 1 on page 329 of your textbook. It can also be done without using calculus if you begin with the result of this example. Try both ways!]

$V = \int_0^h (2x)^2\,dy$ (+1) (inc limits)





$= \frac{4}{4}\int_0^h \left(a - \frac{a-b}{h}y\right)^2 dy$ (+1)

$= \int_0^h \left(a^2 - \frac{2a(a-b)}{h}y + \frac{(a-b)^2}{h^2}y^2\right) dy$ (+1)

$= \left[a^2y - \frac{2a(a-b)}{2h}y^2 + \frac{(a-b)^2}{3h^2}y^3\right]_0^h$ (+1)

$= \left[a^2h - \frac{a-b}{h}ah^2 + \frac{(a-b)^2}{3h^2}h^3\right]$

$= \left[a^2h - a(a-b)h + \frac{(a-b)^2h}{3}\right]$

$= h\left[\cancel{a^2} - \cancel{a^2} + ab + \frac{a^2}{3} - \frac{2ab}{3} + \frac{b^2}{3}\right]$

$= h\left[\frac{a^2}{3} + \frac{ab}{3} + \frac{b^2}{3}\right]$

$= \frac{1}{3}h\left[a^2 + ab + b^2\right]$ (+3)

$\frac{b/2}{c-h} = \frac{a/2}{c}$

$cb = ac - ah$

$c = \frac{-ah}{b-a} = \frac{ah}{a-b}$ (+2)

$\frac{x = \frac{s}{2}}{c-y} = \frac{a/2}{c} = \frac{a(a-b)}{2ah}$

$x = \frac{\frac{1}{2}(a-b)(c-y)}{h}$ $\quad s = \frac{a-b}{h}\left(\frac{ah}{a-b} - y\right)$

$= \frac{\frac{1}{2}(a-b)}{h}\left(\frac{ah}{a-b} - y\right)$ $\quad = a - \frac{a-b}{h}y$

$= \frac{1}{2}\left[a - \frac{a-b}{h}y\right]$ (+2)

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$\frac{14}{68}$