Math 262 Calculus II Lab 6 Arc Length and Surface Area Name: KEY

1. Find the arc length of the graph of the following equations from A to B.

(a) $y=\frac{1}{3}(x^2+2)^{\frac{3}{2}}$; $A(0,\frac{2}{3}\sqrt{2})$ $B(3,\frac{11}{3}\sqrt{11})$

$$\frac{dy}{dx} = \frac{1}{2}(x^2+2)^{\frac{1}{2}}(2x) = x\sqrt{x^2+2}$$

$$L = \int_0^3 \sqrt{1+x^2(x^2+2)}\,dx = \int_0^3 \sqrt{x^4+2x^2+1}\,dx = \int_0^3 \sqrt{(x^2+1)}\,dx$$

$$= \int_0^3 (x^2+1)\,dx = \frac{1}{3}x^3+x\Big|_0^3 = 9+3-0-0 = 12$$

(b) $x=\frac{y^3}{3}+\frac{1}{4y}$; $A(\frac{7}{12},1)$ $B(\frac{109}{12},3)$

$$\frac{dx}{dy} = y^2 - \frac{1}{4y^2}$$

$$L = \int_1^3 \sqrt{1+y^4-\frac{y^2}{2y^2}+\frac{1}{16y^4}}\,dy = \int_1^3 \sqrt{y^4+\frac{1}{2}+\frac{1}{16y^4}}\,dy = \int_1^3 \sqrt{\left(y^2+\frac{1}{4y^2}\right)^2}\,dy$$

$$= \int_1^3 \left(y^2+\frac{1}{4y^2}\right)dy = \frac{1}{3}y^3 - \frac{1}{4y}\Big|_1^3 = 9 - \frac{1}{12} - \frac{1}{3} + \frac{1}{4} = \frac{108-1-4+3}{12} = \frac{106}{12} = \frac{53}{6}$$

(c) $y=\int_0^x \sqrt{\sec^4 t-1}\,dt$; $A(-\frac{\pi}{4},-0.54\ldots)$ $B(\frac{\pi}{4},0.54\ldots)$

$$\frac{dy}{dx} = \sqrt{\sec^4(x)-1}$$

$$L = \int_{-\pi/4}^{\pi/4} \sqrt{1+\sec^4(x)-1}\,dx = \int_{-\pi/4}^{\pi/4} \sec^2(x)\,dx = \tan(x)\Big|_{-\pi/4}^{\pi/4} = 1-(-1) = 2$$

Page 1 SHOW ALL WORK FOR CREDIT GIVE EXACT ANSWERS UNLESS OTHERWISE NOTED

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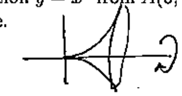
2. The graph of the equation $y=x^3$ from $A(0,0)$ to $B(2,8)$ is revolved about the x-axis. Find the area of the resulting surface.



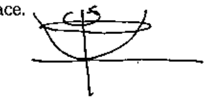
$$\frac{dy}{dx} = 3x^2$$

$$SA = \int_0^2 2\pi(x^3)\sqrt{1+9x^4}\,dx \qquad u=1+9x^4,\ du=36x^3dx$$

$$= \frac{2\pi}{36}\int_1^{145} u^{1/2}\,du = \frac{\pi}{18}\cdot\frac{2}{3}u^{3/2}\Big|_1^{145}$$

$$= \frac{2}{3}\frac{\pi}{18}\left((\sqrt{145})^3-1\right) = \frac{\pi}{27}\left[145\sqrt{145}-1\right]$$

3. The graph of the equation $y=x^2$ from $A(0,0)$ to $B(\sqrt{2},2)$ is revolved about the y-axis. Find the area of the resulting surface.



$$\frac{dy}{dx} = 2x$$

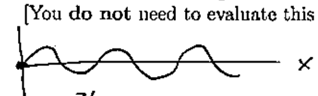
$$SA = \int_0^{\sqrt{2}} 2\pi x\sqrt{1+4x^2}\,dx = \frac{2\pi}{8}\cdot\frac{2}{3}(1+4x^2)^{3/2}\Big|_0^{\sqrt{2}} = \frac{\pi}{6}\left(9^{3/2}-1\right) = \frac{\pi}{6}(26) = \frac{13\pi}{3}$$

or $x=\sqrt{y}$, $\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$

$$SA = \int_0^2 2\pi\sqrt{y}\sqrt{1+\frac{1}{4y}}\,dy = 2\pi\int_0^2\left(y+\frac{1}{4}\right)^{1/2}dy = 2\pi\cdot\frac{2}{3}\left(\sqrt{y+\frac{1}{4}}\right)^3\Big|_0^2$$

$$= \frac{4\pi}{3}\left(\left(\frac{9}{4}\right)^{3/2}-\left(\frac{1}{4}\right)^{3/2}\right) = \frac{4\pi}{3}\left(\frac{26}{8}\right) = \frac{13\pi}{3}$$

4. Set up the integral that will solve the following problem: A manufacturer needs to make corrugated metal sheets 36 inches wide with cross sections in the shape of the curve $y=\frac{1}{2}\sin\pi x$, $0\le x\le 36$. How wide must the original flat sheets be for the manufacturer to produce these corrugated sheets? [You **do not** need to evaluate this integral.]



$$y = \frac{1}{2}\sin(\pi x) \qquad y' = \frac{\pi}{2}\cos(\pi x)$$

$$L = \int_0^{36}\sqrt{1+\frac{\pi^2}{4}\cos^2(\pi x)}\,dx$$

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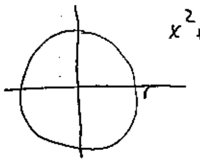
5. Derive a formula for the surface area of a sphere of radius r.



$x^2+y^2=r^2 \qquad y=\sqrt{r^2-x^2} \qquad \frac{dy}{dx} = \frac{-x}{\sqrt{r^2-x^2}}$

$$SA = 2\int_0^r 2\pi\sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$$

$$= 4\pi\int_0^r \sqrt{r^2-x^2+x^2}\,dx$$

$$= 4\pi\int_0^r \sqrt{r^2}\,dx$$

$$= 4\pi r\, x\Big|_0^r$$

$$= 4\pi r^2$$

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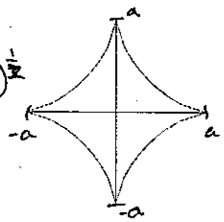
6. An astroid is a curve with an equation of the form $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, for some fixed positive number a. An astroid is traced by a point on a circle of radius $\frac{a}{4}$ that is rolled on the inside of a circle of radius a. The graph is shown below. It was first studied by Johann Bernoulli in 1691.

(a) Find the total length of the astroid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.



$y=\left(a^{\frac{2}{3}}-x^{\frac{2}{3}}\right)^{3/2}$

$\frac{dy}{dx} = \frac{3}{2}\left(a^{\frac{2}{3}}-x^{\frac{2}{3}}\right)^{\frac{1}{2}}\left(-\frac{2}{3}x^{-\frac{1}{3}}\right) = -\frac{1}{x^{1/3}}\left(a^{2/3}-x^{2/3}\right)^{\frac{1}{2}}$

$$L = 4\int_0^a \sqrt{1+\frac{1}{x^{2/3}}\left(a^{2/3}-x^{2/3}\right)}\,dx$$

$$= 4\int_0^a \sqrt{1+a^{2/3}x^{-2/3}-1}\,dx = 4\int_0^a x^{-\frac{1}{3}}\,dx$$

$$= 4a^{1/3}\frac{3}{2}x^{2/3}\Big|_0^a = 6a^{1/3}\left(a^{2/3}-0\right) = 6a$$

$=$

(b) Find the area of the surface generated by revolving the astroid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ about the x-axis.

$$SA = 2\int_0^a 2\pi\left(a^{\frac{2}{3}}-x^{\frac{2}{3}}\right)^{\frac{3}{2}}\sqrt{1+\frac{1}{x^{2/3}}\left(a^{2/3}-x^{2/3}\right)}\,dx$$

$$= 4\pi a^{1/3}\int_0^a x^{-\frac{1}{3}}\left(a^{2/3}-x^{2/3}\right)^{3/2}dx \qquad u=a^{2/3}-x^{2/3} \quad du=-\frac{2}{3}x^{-1/3}dx$$

$$= 4\pi a^{1/3}\left(-\frac{3}{2}\right)\int_{a^{2/3}}^0 u^{3/2}\,du$$

$$= -6\pi a^{1/3}\,\frac{2}{5}u^{5/2}\Big|_{a^{2/3}}^0$$

$$= 0+\frac{12\pi}{5}a^{1/3}\left(a^{2/3}\right)^{5/2} = \frac{12\pi}{5}a^2$$

(c) Look carefully at the integrals that you evaluated in the two parts above. Is there a problem with either of these integrals? Can you think of a way to avoid this difficulty? We will learn more about this problem in chapter 10.

Both have division by 0 at an endpoint of the interval of integration.

Avoid the problem by using limits.