5.1) 6, 10, 32

6) Prove $1\cdot 1! + 2\cdot 2! + \cdots + n\cdot n! = (n+1)! - 1$

Pf Basis: If $n=1$, this says

$$1\cdot 1! = 2! - 1$$

or $1 = 1$

which is obviously true.

Inductive step

Assume $1\cdot 1! + \cdots + k\cdot k! = (k+1)! - 1$

we want to show

$$1\cdot 1! + \cdots + k\cdot k! + (k+1)\cdot(k+1)! = (k+2)! - 1.$$

But

$$1\cdot 1! + \cdots + k\cdot k! + (k+1)\cdot(k+1)!$$

$$= (k+1)! - 1 + (k+1)(k+1)! \quad \text{by ind've hyp}$$

$$= (k+1)![1 + (k+1)] - 1$$

$$= (k+1)!(k+2) - 1$$

$$= (k+2)! - 1.$$

So $1\cdot 1! + \cdots + n\cdot n!$ is true $\forall n \in \mathbb{Z}^+$

10) a) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$

$n=1 \rightarrow \frac{1}{2}$

$n=2 \quad \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$

$n=3 \quad \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{6+2+1}{12} = \frac{3}{4}$

Conjecture: equals $\frac{n}{n+1}$

Pf Basis: If $n=1$, this says

$\frac{1}{1\cdot 2} = \frac{1}{2}$, which is true

Inductive:

Assume $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

Then

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \quad \text{by Ind've Hyp}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2},$$

which is what we wanted to show.

So $\frac{1}{1\cdot 2} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1} \quad \forall n \in \mathbb{Z}^+$ by PMI.

32) $3 \mid (n^3+2n)$

Pf Basis: If $n=1$, then $n^3+2n = 3$, and $3 \mid 3$.

Ind've: Assume $3 \mid (k^3+2k)$. Then

$$(k+1)^3 + 2(k+1) =$$

$$= k^3 + 3k^2 + 3k + 1 + 2k + 2$$

$$= (k^3+2k) + (3k^2+3k+3)$$

$$= (k^3+2k) + 3(k^2+k+1)$$

$3 \mid (k^3+2k)$ by ind've hyp. And $3 \mid [3(k^2+k+1)]$ since k^2+k+1 is $\in \mathbb{Z}$

So $3 \mid (n^3+2n) \quad \forall n \in \mathbb{Z}^+$ by PMI. ∎