# An introduction to the Jacobian conjecture

Damiano Fulghesu

Minnesota State University Moorhead

October 12, 2010

# Origin of the conjecture.

First formulated by O. Keller in 1939.

# Origin of the conjecture.

First formulated by O. Keller in 1939.

Hartshorne's Exercise 3.19 (b).

# Origin of the conjecture.

First formulated by O. Keller in 1939.

Hartshorne's Exercise 3.19 (b).

It is one of the 18 Smale's problems.

# Origin of the conjecture.

First formulated by O. Keller in 1939.

Hartshorne's Exercise 3.19 (b).

It is one of the 18 Smale's problems.

Still unproven?

**Base field $\mathbb{C}$.**

# Polynomial maps and their Jacobian

**Base field $\mathbb{C}$.**

## Polynomial map

$$\begin{aligned} F = (F_1, \ldots, F_n) : \mathbb{C}^n &\rightarrow \mathbb{C}^n \\ \underline{z} = (z_1, \ldots, z_n) &\mapsto (F_1(\underline{z}), \ldots, F_n(\underline{z})) \end{aligned}$$

# Polynomial maps and their Jacobian

**Base field $\mathbb{C}$.**

## Polynomial map

$$\begin{aligned} F = (F_1, \ldots, F_n) : \mathbb{C}^n &\to \mathbb{C}^n \\ \underline{z} = (z_1, \ldots, z_n) &\mapsto (F_1(\underline{z}), \ldots, F_n(\underline{z})) \end{aligned}$$

## Jacobian

$$J_F(\underline{z}) = \begin{pmatrix} \frac{\partial F_1}{\partial z_1}(\underline{z}) & \ldots & \frac{\partial F_1}{\partial z_n}(\underline{z}) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial z_1}(\underline{z}) & \ldots & \frac{\partial F_n}{\partial z_n}(\underline{z}) \end{pmatrix}$$

# Invertible polynomial maps

# Invertible polynomial maps

**Theorem**

If $F : \mathbb{C}^n \to \mathbb{C}^n$ is invertible (and its inverse is a polynomial map) **then**

$$|J_F(\underline{z})| \in \mathbb{C}^*$$

(the determinant of $J_F(\underline{z})$ is a nonzero constant)

# Invertible polynomial maps

**Theorem**

If $F : \mathbb{C}^n \to \mathbb{C}^n$ is invertible (and its inverse is a polynomial map) **then**

$$|J_F(\underline{z})| \in \mathbb{C}^*$$

(the determinant of $J_F(\underline{z})$ is a nonzero constant)

**Proof:** Let $G : \mathbb{C}^n \to \mathbb{C}^n$ be the inverse of $F$.

# Invertible polynomial maps

**Theorem**

If $F : \mathbb{C}^n \to \mathbb{C}^n$ is invertible (and its inverse is a polynomial map) **then**

$$|J_F(\underline{z})| \in \mathbb{C}^*$$

(the determinant of $J_F(\underline{z})$ is a nonzero constant)

**Proof:** Let $G : \mathbb{C}^n \to \mathbb{C}^n$ be the inverse of $F$. We must have that the compositions $F \circ G$ and $G \circ F$ are the identity maps.

# Invertible polynomial maps

**Theorem**

If $F : \mathbb{C}^n \to \mathbb{C}^n$ is invertible (and its inverse is a polynomial map) **then**

$$|J_F(\underline{z})| \in \mathbb{C}^*$$

(the determinant of $J_F(\underline{z})$ is a nonzero constant)

**Proof:** Let $G : \mathbb{C}^n \to \mathbb{C}^n$ be the inverse of $F$. We must have that the compositions $F \circ G$ and $G \circ F$ are the identity maps. By applying the **chain rule**, we get

$$J_{G\circ F}(\underline{z}) = J_G(F(\underline{z})) \cdot J_F(\underline{z}) = I_n$$

# Invertible polynomial maps

**Theorem**

If $F : \mathbb{C}^n \to \mathbb{C}^n$ is invertible (and its inverse is a polynomial map) **then**

$$|J_F(\underline{z})| \in \mathbb{C}^*$$

(the determinant of $J_F(\underline{z})$ is a nonzero constant)

**Proof:** Let $G : \mathbb{C}^n \to \mathbb{C}^n$ be the inverse of $F$. We must have that the compositions $F \circ G$ and $G \circ F$ are the identity maps. By applying the **chain rule**, we get

$$J_{G \circ F}(\underline{z}) = J_G(F(\underline{z})) \cdot J_F(\underline{z}) = I_n$$

In particular, for every $\underline{z}$, the determinant $|J_F(\underline{z})|$ must be different from 0.

# Invertible polynomial maps

**Theorem**

If $F : \mathbb{C}^n \to \mathbb{C}^n$ is invertible (and its inverse is a polynomial map)
**then**

$$|J_F(\underline{z})| \in \mathbb{C}^*$$

(the determinant of $J_F(\underline{z})$ is a nonzero constant)

**Proof:** Let $G : \mathbb{C}^n \to \mathbb{C}^n$ be the inverse of $F$. We must have that the compositions $F \circ G$ and $G \circ F$ are the identity maps. By applying the **chain rule**, we get

$$J_{G\circ F}(\underline{z}) = J_G(F(\underline{z})) \cdot J_F(\underline{z}) = I_n$$

In particular, for every $\underline{z}$, the determinant $|J_F(\underline{z})|$ must be different from 0.
We conclude by observing that $J_F(\underline{z})$ is a **polynomial** and $\mathbb{C}$ is **algebraically closed**.

# The Jacobian conjecture

# The Jacobian conjecture

**Conjecture**

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that

$$|J_F(\underline{z})| \in \mathbb{C}^*$$

**then**

$F$ is invertible (and its inverse is a polynomial map).

# The Jacobian conjecture

**Conjecture**

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that

$$|J_F(\underline{z})| \in \mathbb{C}^*$$

**then**

$F$ is invertible (and its inverse is a polynomial map).

Still open for $n \geq 2$!!!

**Conjecture**

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that

$$|J_F(\underline{z})| \in \mathbb{C}^*$$

**then**

$F$ is invertible (and its inverse is a polynomial map).

Still open for $n \geq 2$!!!

**More generally**

Instead of $\mathbb{C}$, we can consider any algebraically closed field $k$ with characteristic 0.

# The case $n = 1$

**The conjecture is true for $n = 1$**

**The conjecture is true for $n = 1$**

The map is a polynomial $F(z)$, such that

$$\frac{\partial F}{\partial z}(z) = a \neq 0$$

**The conjecture is true for $n = 1$**

The map is a polynomial $F(z)$, such that

$$\frac{\partial F}{\partial z}(z) = a \neq 0$$

Therefore

**The conjecture is true for $n = 1$**

The map is a polynomial $F(z)$, such that

$$\frac{\partial F}{\partial z}(z) = a \neq 0$$

Therefore

- $F(z) = az + b$ with $a \neq 0$

**The conjecture is true for $n = 1$**

The map is a polynomial $F(z)$, such that

$$\frac{\partial F}{\partial z}(z) = a \neq 0$$

Therefore

- $F(z) = az + b$ with $a \neq 0$
- the inverse is $G(z) = \frac{z-b}{a}$

# An example for $n = 2$

# An example for $n = 2$

We define

$$\begin{aligned} F : \mathbb{C}^2 &\to \mathbb{C}^2 \\ (z_1, z_2) &\mapsto \left(z_1 + (z_1 + z_2)^3, z_2 - (z_1 + z_2)^3\right) \end{aligned}$$

# An example for $n = 2$

We define

$$\begin{aligned} F : \mathbb{C}^2 &\rightarrow \mathbb{C}^2 \\ (z_1, z_2) &\mapsto \left(z_1 + (z_1 + z_2)^3, z_2 - (z_1 + z_2)^3\right) \end{aligned}$$

We have (Exercise):

# An example for $n = 2$

We define

$$\begin{aligned} F : \mathbb{C}^2 &\rightarrow \mathbb{C}^2 \\ (z_1, z_2) &\mapsto \left(z_1 + (z_1 + z_2)^3, z_2 - (z_1 + z_2)^3\right) \end{aligned}$$

We have (Exercise):

- the determinant of $J_F(z_1, z_2)$ is 1

# An example for $n = 2$

We define

$$\begin{aligned} F : \mathbb{C}^2 &\rightarrow \mathbb{C}^2 \\ (z_1, z_2) &\mapsto \left(z_1 + (z_1 + z_2)^3, z_2 - (z_1 + z_2)^3\right) \end{aligned}$$

We have (Exercise):

- the determinant of $J_F(z_1, z_2)$ is 1
- the inverse of $F$ is

$$\begin{aligned} G : \mathbb{C}^2 &\rightarrow \mathbb{C}^2 \\ (z_1, z_2) &\mapsto \left(z_1 - (z_1 + z_2)^3, z_2 + (z_1 + z_2)^3\right) \end{aligned}$$

# Linear Case

# Linear Case

**Definition**

$$\deg(F) = \max_i \deg(F_i)$$

# Linear Case

**Definition**

$$\deg(F) = \max_i \deg(F_i)$$

Up to replacing $F(\underline{z})$ with $F(\underline{z}) - F(0)$, we can assume

$$F(0) = 0$$

# Linear Case

**Definition**

$$\deg(F) = \max_i \deg(F_i)$$

Up to replacing $F(\underline{z})$ with $F(\underline{z}) - F(0)$, we can assume

$$F(0) = 0$$

and we consider the decomposition in homogenous components

$$F = F^{(1)} + \cdots + F^{(d)}$$

**Definition**

$$\deg(F) = \max_i \deg(F_i)$$

Up to replacing $F(\underline{z})$ with $F(\underline{z}) - F(0)$, we can assume

$$F(0) = 0$$

and we consider the decomposition in homogenous components

$$F = F^{(1)} + \cdots + F^{(d)}$$

**Theorem (Linear Algebra)**

If $\deg(F) = 1$ then the Jacobian conjecture is true.

# Reduction to injectivity

# Reduction to injectivity

**A. Białynicki-Birula, M. Rosenlicht (1962)**

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. If $F$ is injective, then $F$ is surjective.

# Reduction to injectivity

**A. Białynicki-Birula, M. Rosenlicht (1962)**

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. If $F$ is injective, then $F$ is surjective.

**S. Cynk, K. Rusek (1991)**

If $F : \mathbb{C}^n \to \mathbb{C}^n$ is a bijective polynomial map, then the inverse of $F$ is a polynomial map.

# Reduction to injectivity

**A. Białynicki-Birula, M. Rosenlicht (1962)**

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. If $F$ is injective, then $F$ is surjective.

**S. Cynk, K. Rusek (1991)**

If $F : \mathbb{C}^n \to \mathbb{C}^n$ is a bijective polynomial map, then the inverse of $F$ is a polynomial map.

therefore the Jacobian conjecture reduces to prove that the polynomial map is injective

# Reduction to injectivity

**A. Białynicki-Birula, M. Rosenlicht (1962)**

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. If $F$ is injective, then $F$ is surjective.

**S. Cynk, K. Rusek (1991)**

If $F : \mathbb{C}^n \to \mathbb{C}^n$ is a bijective polynomial map, then the inverse of $F$ is a polynomial map.

therefore the Jacobian conjecture reduces to prove that the polynomial map is injective

**More generally**

Cynk and Rusek proved that if $V$ is an affine algebraic set over an algebraically closed field $k$ of characteristic 0 and $F : V \to V$ is an injective endomorphism, then $F$ is an automorphism.

## Quadratic case

First proved by S. Wang in 1980

# Quadratic case

First proved by S. Wang in 1980

1. Let us suppose that $\deg F \leq 2$ and $|J_F(\underline{z})| \in \mathbb{C}^*$

# Quadratic case

First proved by S. Wang in 1980

1. Let us suppose that $\deg F \leq 2$ and $|J_F(\underline{z})| \in \mathbb{C}^*$
2. Suppose that $F(\underline{a}) = F(\underline{b})$ for some $\underline{a} \neq \underline{b}$

# Quadratic case

First proved by S. Wang in 1980

1. Let us suppose that $\deg F \leq 2$ and $|J_F(\underline{z})| \in \mathbb{C}^*$
2. Suppose that $F(\underline{a}) = F(\underline{b})$ for some $\underline{a} \neq \underline{b}$

Replace $F(\underline{z})$ with $F(\underline{z} + \underline{a}) - F(\underline{a})$ and consider $\underline{c} = \underline{b} - \underline{a}$,
we may assume

First proved by S. Wang in 1980

1. Let us suppose that $\deg F \leq 2$ and $|J_F(\underline{z})| \in \mathbb{C}^*$
2. Suppose that $F(\underline{a}) = F(\underline{b})$ for some $\underline{a} \neq \underline{b}$

Replace $F(\underline{z})$ with $F(\underline{z} + \underline{a}) - F(\underline{a})$ and consider $\underline{c} = \underline{b} - \underline{a}$, we may assume

- $F(0) = 0$

# Quadratic case

First proved by S. Wang in 1980

1. Let us suppose that $\deg F \leq 2$ and $|J_F(\underline{z})| \in \mathbb{C}^*$
2. Suppose that $F(\underline{a}) = F(\underline{b})$ for some $\underline{a} \neq \underline{b}$

Replace $F(\underline{z})$ with $F(\underline{z} + \underline{a}) - F(\underline{a})$ and consider $\underline{c} = \underline{b} - \underline{a}$, we may assume

- $F(0) = 0$
- $F(\underline{c}) = 0$ for some $\underline{c} \neq 0$

# Quadratic case

First proved by S. Wang in 1980

1. Let us suppose that $\deg F \leq 2$ and $|J_F(\underline{z})| \in \mathbb{C}^*$
2. Suppose that $F(\underline{a}) = F(\underline{b})$ for some $\underline{a} \neq \underline{b}$

Replace $F(\underline{z})$ with $F(\underline{z}+\underline{a}) - F(\underline{a})$ and consider $\underline{c} = \underline{b} - \underline{a}$, we may assume

- $F(0) = 0$
- $F(\underline{c}) = 0$ for some $\underline{c} \neq 0$

Let us write

$$F(\underline{z}) = F^{(1)}(\underline{z}) + F^{(2)}(\underline{z})$$

# Quadratic case

First proved by S. Wang in 1980

1. Let us suppose that deg $F \leq 2$ and $|J_F(\underline{z})| \in \mathbb{C}^*$
2. Suppose that $F(\underline{a}) = F(\underline{b})$ for some $\underline{a} \neq \underline{b}$

Replace $F(\underline{z})$ with $F(\underline{z} + \underline{a}) - F(\underline{a})$ and consider $\underline{c} = \underline{b} - \underline{a}$,
we may assume

- $F(0) = 0$
- $F(\underline{c}) = 0$ for some $\underline{c} \neq 0$

Let us write

$$F(\underline{z}) = F^{(1)}(\underline{z}) + F^{(2)}(\underline{z})$$

We have, for all $t \in \mathbb{C}$,

$$F(t\underline{c}) = tF^{(1)}(\underline{c}) + t^2 F^{(2)}(\underline{c})$$

# Quadratic case

1. Let us suppose that $\deg F \leq 2$ and $|J_F(\underline{z})| \in \mathbb{C}^*$
2. Suppose that $F(\underline{c}) = F(0) = 0$ for some $\underline{c} \neq 0$

We differentiate and we get, for all $t \in \mathbb{C}$,

$$\frac{\partial}{\partial t} F(t\underline{c}) = F^{(1)}(\underline{c}) + 2tF^{(2)}(\underline{c}) = J_F(t\underline{c}) \cdot \underline{c} \neq 0$$

# Quadratic case

1. Let us suppose that $\deg F \leq 2$ and $|J_F(\underline{z})| \in \mathbb{C}^*$
2. Suppose that $F(\underline{c}) = F(0) = 0$ for some $\underline{c} \neq 0$

We differentiate and we get, for all $t \in \mathbb{C}$,

$$\frac{\partial}{\partial t} F(t\underline{c}) = F^{(1)}(\underline{c}) + 2tF^{(2)}(\underline{c}) = J_F(t\underline{c}) \cdot \underline{c} \neq 0$$

in particular, when $t = \frac{1}{2}$,

$$F(\underline{c}) = F^{(1)}(\underline{c}) + F^{(2)}(\underline{c}) = J_F\left(\frac{1}{2}\underline{c}\right) \cdot \underline{c} \neq 0$$

**contradiction.**

# Counter-example with analytic functions

# Counter-example with analytic functions

Consider

$$\begin{aligned} F : \mathbb{C} &\rightarrow \mathbb{C} \\ z &\mapsto e^z \end{aligned}$$

# Counter-example with analytic functions

Consider

$$\begin{aligned} F : \mathbb{C} &\rightarrow \mathbb{C} \\ z &\mapsto e^z \end{aligned}$$

We have

- $J_F(z) = e^z \neq 0$

# Counter-example with analytic functions

Consider

$$\begin{aligned} F : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto e^z \end{aligned}$$

We have

- $J_F(z) = e^z \neq 0$
- $F$ is not injective, because, over $\mathbb{C}$, the function $e^z$ is periodic with period $2\pi i$.

# Counter-example with real numbers

# Counter-example with real numbers

Consider

$$\begin{aligned} f:\mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^3 + x \end{aligned}$$

# Counter-example with real numbers

Consider

$$\begin{aligned} f : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto x^3 + x \end{aligned}$$

We have

- $J_f(x) = f'(x) = 3x^2 + 1 \neq 0$ for all $x$, since we are on $\mathbb{R}$

# Counter-example with real numbers

Consider

$$\begin{aligned} f : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto x^3 + x \end{aligned}$$

We have

- $J_f(x) = f'(x) = 3x^2 + 1 \neq 0$ for all $x$, since we are on $\mathbb{R}$
- $f(x)$ is bijective

# Counter-example with real numbers

Consider

$$\begin{array}{rcl} f:\mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3+x \end{array}$$

We have

- $J_f(x) = f'(x) = 3x^2 + 1 \neq 0$ for all $x$, since we are on $\mathbb{R}$
- $f(x)$ is bijective
- the inverse of $f(x)$ cannot be a polynomial (exercise)

# Simplifying the linear part

# Simplifying the linear part

**Fact**

We can reduce to the case

$$F(\underline{z}) = I_n \cdot \underline{z} + F^{(2)}(\underline{z}) + \cdots + F^{(d)}(\underline{z})$$

# Simplifying the linear part

**Fact**

We can reduce to the case

$$F(\underline{z}) = I_n \cdot \underline{z} + F^{(2)}(\underline{z}) + \cdots + F^{(d)}(\underline{z})$$

**Proof** Let us write

$$F(\underline{z}) = F^{(1)}(\underline{z}) + F^{(2)}(\underline{z}) + \cdots + F^{(d)}(\underline{z})$$

# Simplifying the linear part

**Fact**

We can reduce to the case

$$F(\underline{z}) = I_n \cdot \underline{z} + F^{(2)}(\underline{z}) + \cdots + F^{(d)}(\underline{z})$$

**Proof** Let us write

$$F(\underline{z}) = F^{(1)}(\underline{z}) + F^{(2)}(\underline{z}) + \cdots + F^{(d)}(\underline{z})$$

We have

$$J_F(\underline{z}) = J_{F^{(1)}} + J_{F^{(2)}}(\underline{z}) + \cdots + J_{F^{(d)}}(\underline{z})$$

# Simplifying the linear part

**Fact**

We can reduce to the case

$$F(\underline{z}) = I_n \cdot \underline{z} + F^{(2)}(\underline{z}) + \cdots + F^{(d)}(\underline{z})$$

**Proof** Let us write

$$F(\underline{z}) = F^{(1)}(\underline{z}) + F^{(2)}(\underline{z}) + \cdots + F^{(d)}(\underline{z})$$

We have

$$J_F(\underline{z}) = J_{F^{(1)}} + J_{F^{(2)}}(\underline{z}) + \cdots + J_{F^{(d)}}(\underline{z})$$

In particular

$$J_F(0) = J_{F^{(1)}}$$

# Simplifying the linear part

**Fact**

We can reduce to the case

$$F(\underline{z}) = I_n \cdot \underline{z} + F^{(2)}(\underline{z}) + \cdots + F^{(d)}(\underline{z})$$

**Proof** Let us write

$$F(\underline{z}) = F^{(1)}(\underline{z}) + F^{(2)}(\underline{z}) + \cdots + F^{(d)}(\underline{z})$$

We have

$$J_F(\underline{z}) = J_{F^{(1)}} + J_{F^{(2)}}(\underline{z}) + \cdots + J_{F^{(d)}}(\underline{z})$$

In particular

$$J_F(0) = J_{F^{(1)}}$$

and since the determinant of the Jacobian is supposed to be a constant

$$|J_F(\underline{z})| = |J_{F^{(1)}}|$$

# Simplifying the linear part

**Fact**

We can reduce to the case

$$F(\underline{z}) = I_n \cdot \underline{z} + F^{(2)}(\underline{z}) + \cdots + F^{(d)}(\underline{z})$$

In particular we have that the matrix $J_{F^{(1)}}$ is invertible

# Simplifying the linear part

**Fact**

We can reduce to the case

$$F(\underline{z}) = I_n \cdot \underline{z} + F^{(2)}(\underline{z}) + \cdots + F^{(d)}(\underline{z})$$

In particular we have that the matrix $J_{F^{(1)}}$ is invertible

After a linear change of coordinates we can assume

$$J_{F^{(1)}} = I_n$$

# Reduction of degree

# Reduction of degree

**A. Yagzhev (1980); H. Bass, E. Connell, D. Wright (1982)**

If the Jacobian conjecture holds for all $n \geq 2$ and all $F : \mathbb{C}^n \to \mathbb{C}^n$ of the form

$$F = I_n + F^{(3)}$$

then the Jacobian conjecture holds.

# Reduction of degree

**A. Yagzhev (1980); H. Bass, E. Connell, D. Wright (1982)**

If the Jacobian conjecture holds for all $n \geq 2$ and all $F : \mathbb{C}^n \to \mathbb{C}^n$ of the form

$$F = I_n + F^{(3)}$$

then the Jacobian conjecture holds.

**Warning**

This **does not mean**, for example, that if we prove the conjecture for $F = I_n + F^{(3)}$ in the case $n = 2$, then we have proved the conjecture for the case $n = 2$.

# Reduction of degree

**A. Yagzhev (1980); H. Bass, E. Connell, D. Wright (1982)**

If the Jacobian conjecture holds for all $n \geq 2$ and all $F : \mathbb{C}^n \to \mathbb{C}^n$ of the form

$$F = I_n + F^{(3)}$$

then the Jacobian conjecture holds.

**Warning**

This **does not mean**, for example, that if we prove the conjecture for $F = I_n + F^{(3)}$ in the case $n = 2$, then we have proved the conjecture for the case $n = 2$.

**M. de Bondt, A. van den Essen (2005)**

If the Jacobian conjecture holds for all $F = I_n + F^{(3)}$ such that $J_F$ is symmetric, then the Jacobian conjecture holds.

# Reduction of degree

# Reduction of degree

**L.M. Drużkowski (1983)**

If the Jacobian conjecture holds for all $n \geq 2$ and all $F : \mathbb{C}^n \to \mathbb{C}^n$ of the form

$$F(\underline{z}) = I_n \cdot \underline{z} + \left( \left( \sum_{k=1}^{n} a_{k,1} z_k \right)^3, \ldots, \left( \sum_{k=1}^{n} a_{k,n} z_k \right)^3 \right)$$

then the Jacobian conjecture holds.

# Reduction of degree

**L.M. Drużkowski (1983)**

If the Jacobian conjecture holds for all $n \geq 2$ and all $F : \mathbb{C}^n \to \mathbb{C}^n$ of the form

$$F(\underline{z}) = I_n \cdot \underline{z} + \left( \left( \sum_{k=1}^{n} a_{k,1} z_k \right)^3, \ldots, \left( \sum_{k=1}^{n} a_{k,n} z_k \right)^3 \right)$$

then the Jacobian conjecture holds.

**E. Hubbers (1994)**

The Jacobian conjecture holds for all $F$ of the Drużkowski form if $n \leq 7$.

# Reduction of degree

**L.M. Drużkowski (1983)**

If the Jacobian conjecture holds for all $n \geq 2$ and all $F : \mathbb{C}^n \to \mathbb{C}^n$ of the form

$$F(\underline{z}) = I_n \cdot \underline{z} + \left( \left( \sum_{k=1}^{n} a_{k,1} z_k \right)^3, \ldots, \left( \sum_{k=1}^{n} a_{k,n} z_k \right)^3 \right)$$

then the Jacobian conjecture holds.

**E. Hubbers (1994)**

The Jacobian conjecture holds for all $F$ of the Drużkowski form if $n \leq 7$.

**M. de Bondt, A. van den Essen (2005)**

The Jacobian conjecture holds for all $F$ of the Drużkowski form such that $J_F$ is symmetric.

# Other results

# Other results

- D. Wright (1993) Jacobian conjecture holds for $n = 3$ and $F = I_n + F^{(3)}$

# Other results

- D. Wright (1993) Jacobian conjecture holds for $n = 3$ and $F = I_n + F^{(3)}$
- E. Hubbers (1994) Jacobian conjecture holds for $n = 4$ and $F = I_n + F^{(3)}$

# Other results

- D. Wright (1993) Jacobian conjecture holds for $n = 3$ and $F = I_n + F^{(3)}$
- E. Hubbers (1994) Jacobian conjecture holds for $n = 4$ and $F = I_n + F^{(3)}$
- T. Moh (1983) Jacobian conjecture holds for $n = 2$ and $d \leq 100$

# Other results

- D. Wright (1993) Jacobian conjecture holds for $n = 3$ and $F = I_n + F^{(3)}$
- E. Hubbers (1994) Jacobian conjecture holds for $n = 4$ and $F = I_n + F^{(3)}$
- T. Moh (1983) Jacobian conjecture holds for $n = 2$ and $d \leq 100$
- L. Wang (2005) found more exceptional cases than Moh and confirmed the theorem

# Other results

- D. Wright (1993) Jacobian conjecture holds for $n = 3$ and $F = I_n + F^{(3)}$
- E. Hubbers (1994) Jacobian conjecture holds for $n = 4$ and $F = I_n + F^{(3)}$
- T. Moh (1983) Jacobian conjecture holds for $n = 2$ and $d \leq 100$
- L. Wang (2005) found more exceptional cases than Moh and confirmed the theorem
- M. Razar (1979) Jacobian conjecture holds for $n = 2$ if the all the fibers of $F_1$ or $F_2$ are irreducible rational curves

Thank you!