| Notes Math 127 | |
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*The intent of these notes is to summarize basic definitions and to give some examples that have been done in class. Students are recommended to read the textbook for further examples and facts.*
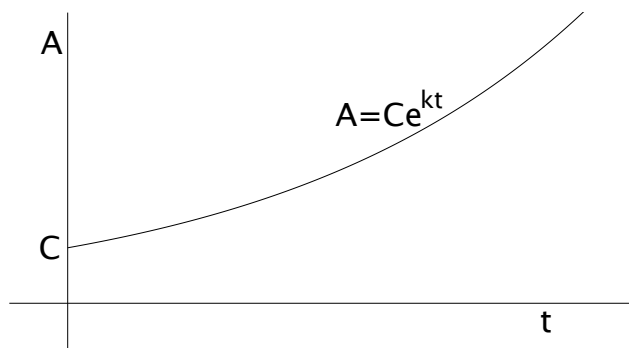
# Exponential Growth and Decay.

# 1 Exponential Models

Every model is, at the very end, an approximation of real phenomena. We will describe some models that are effective only for a limited amount of time.
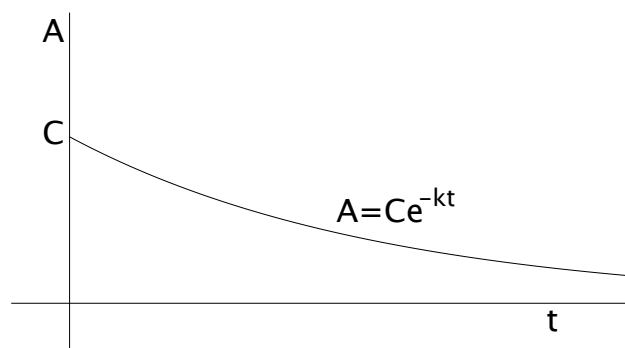
We have an initial amount (of a population or of a sample) $C$. Let $k > 0$ be the growth rate (or the decay rate). The mathematical models for exponential growth or decay are [1]:

| exponential growth | $A = Ce^{kt}$ |
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| exponential decay | $A = Ce^{-kt}$ |

Exponential Growth                                    Exponential Decay



**Example 1.** In 2000, the population of Israel was approximately 6.04 million and by 2050 it is projected to grow to 10 million.

(a) Use the exponential growth model $A = Ce^{kt}$, in which $t$ is the number of years after 2000, to find an exponential growth function that models the data.

(b) What is Israel's population in 2011?

(c) In which year will Israel's population be 9 million?

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[1] We have seen something similar for the interest compounded continuously $A = Pe^{rt}$.

**Solution.**

(a) We are considering 2000 as our initial time, so the initial population is $C = 6.04$, therefore we can write:
$$A = 6.04e^{kt}$$

We need to find $k$, so we plug-in $A = 10$ and $t = 2050 - 2000 = 50$. Then we solve for $k$ (we perform all calculations at the end):

$$
\begin{aligned}
6.04e^{50k} &= 10 \\
e^{50k} &= 10/6.04 \\
50k &= \ln(10/6.04) \\
k &= \frac{\ln(10/6.04)}{50} \approx 0.01
\end{aligned}
$$

The function is $\boxed{A = 6.04e^{0.01t}}$.

(b) We evaluate the function of part (a) at $t = 11$

$A = 6.04e^{0.01(11)} \approx \boxed{6.7}$ million.

(c) We have to solve

$$
\begin{aligned}
6.04e^{0.01t} &= 9 \\
e^{0.01t} &= 9/6.04 \\
0.01t &= \ln(9/6.04) \\
t &= \frac{\ln(9/6.04)}{0.01} \approx \boxed{40}
\end{aligned}
$$

Israel's population will be 9 million in 2040.

$\square$

**Example 2.** An artifact originally had 40 grams of carbon-14 present. The decay model $A = 40e^{-0.000121t}$ describes the amount of carbon-14 present after $t$ years. Now the artifact has 25 grams of carbon-14. How old is the artifact?

**Solution.** We solve

$$
\begin{aligned}
40e^{-0.000121t} &= 25 \\
e^{-0.000121t} &= 25/40 \\
-0.000121t &= \ln(25/40) \\
t &= \frac{\ln(25/40)}{-0.000121} \approx \boxed{3884}
\end{aligned}
$$

The artifact is 3884 years old. $\square$

# 2   Half-Life of Radioactive Substances

The ⟨half-life⟩ of a radioactive substance is the time required for half of a the given substance to decay (and be no more radioactive).

**Example 3.** The half-life of carbon-14 is 5715 years. Find the exponential decay model $A = Ce^{-kt}$ for carbon-14.

**Solution.** We need to find $k$. If $C$ is the original amount of carbon-14, then when $t = 5715$ we have $A = \frac{1}{2}C$. So we can write

$$\frac{1}{2}C = Ce^{-5715k}$$

We simplify $C$ in both sides and we solve for $k$:

$$e^{-5715k} = \frac{1}{2}$$

$$-5715k = \ln\left(\frac{1}{2}\right)$$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{-5715} = \frac{-\ln 2}{-5715} = \frac{\ln 2}{5715} \approx \boxed{0.000121}$$

The exponential decay model for carbon-14 is

$$\boxed{A = Ce^{-0.000121t}}$$

Compare this result with Example **??**                                                   □

**In general:** if the half-life of a substance is $H$, then the exponential decay model is $A = Ce^{-kt}$ where

$$\boxed{k = \frac{\ln 2}{H}}$$

**Example 4.** The half life of radium-226 is 1620 years. A certain sample has 40% of the original amount of radium-226. How old is the sample?

**Solution.** We get $k = \dfrac{\ln 2}{1620} \approx 0.000428$, so the decay model is

$$A = Ce^{-0.000428t}$$

Now, if $C$ is the original amount of radium-226, the 40% of $C$ is $0.4C$. Therefore we have to solve

$$0.4C = Ce^{-000428t}$$

we can simplify $C$ in both sides and we get

$$
\begin{aligned}
e^{-000428t} &= 0.4 \\
-000428t &= \ln(0.4) \\
t &= \frac{\ln(0.4)}{-000428} \approx \boxed{2141}
\end{aligned}
$$

The sample is 2141 years old. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

# 3   Logistic Growth Models

Now we want to consider the fact that there is a limit of growth. A function that models this phenomenon is the $\boxed{\textbf{logistic function}}$:

$$
A(t) = \frac{C}{1 + ae^{-bt}}
$$

where $A(t)$ is the amount at the time $t$, while $a$, $b$, and $C$ are positive constants. In particular $C$ is the maximum amount allowed (since, as $t$ approaches to $\infty$, the value of $e^{-bt}$ approaches to $0$). The initial amount, that is to say the amount when $t = 0$, is given by

$$
A(0) = \frac{C}{1 + ae^{-b \cdot 0}} = \frac{C}{1 + a}
$$

So, in general, we can say that the initial amount for the logistic model is $\boxed{\dfrac{C}{1 + a}}$.

Now you may wonder what $b$ represents. So far you need to know that the larger $b$ is, the faster the population is approaching to the maximum [2].

As an example we give the graph of the function:

$$
A(t) = \frac{15}{1 + 14e^{-0.2t}}
$$

In particular notice that the maximum is 15, while the initial amount is $\dfrac{15}{1 + 14} = 1$. We also remark that though the constant $b = 0.2$ is relatively small, after 35 units of time we are very close to the maximum.

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[2]You will be able to understand this better in your Calculus class.
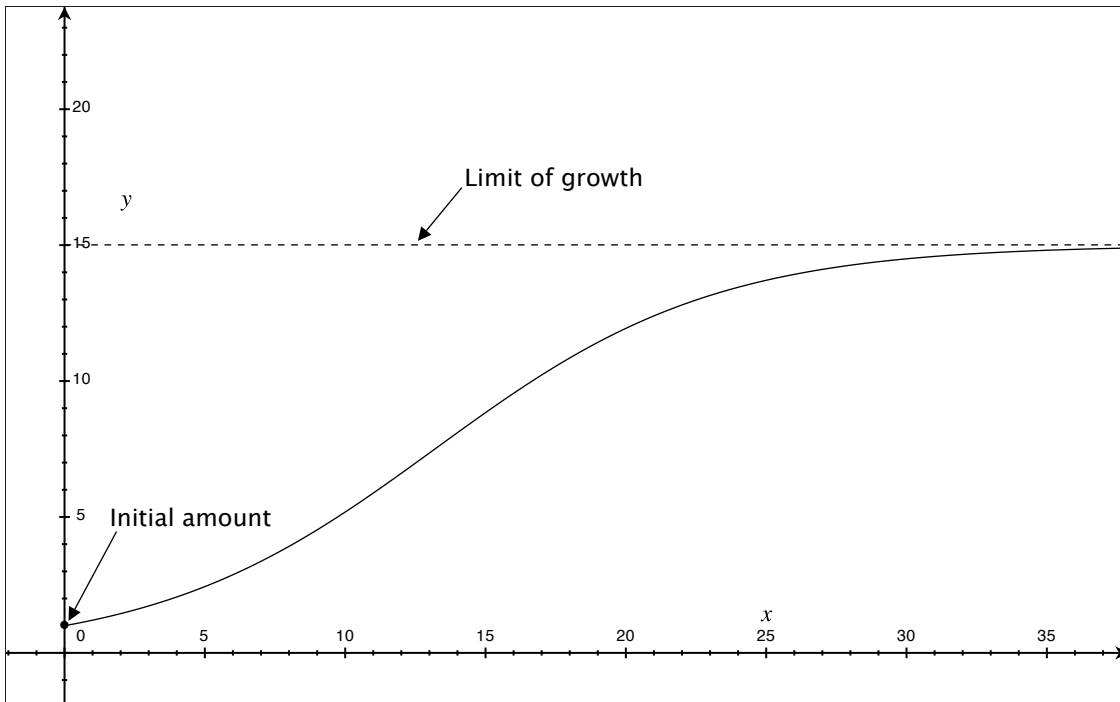
**Example 5.** The logistic growth function $P(x) = \dfrac{90}{1 + 271e^{-0.122x}}$ models the percentage, $P(x)$, of Americans who are $x$ years old with some coronary heart disease.

(a) What is the limiting percentage of $x$-year-olds that have some coronary heart disease?

(b) What percentage of 20-year-olds have some coronary heart disease?

(c) What percentage of 80-year-olds have some coronary heart disease?

(d) At what age is the percentage of some coronary heart disease 50%?

**Solution.**

(a) The limiting percentage is the number appearing at the numerator: $\boxed{90\%}$

(b) We need to evaluate $P(20)$:

$$P(20) = \frac{90}{1 + 271e^{-0.122(20)}} \approx \boxed{3.66\%}.$$

(c) We need to evaluate $P(80)$:

$$P(80) = \frac{90}{1 + 271e^{-0.122(80)}} \approx \boxed{88.61\%}.$$

(d) We need to solve $P(x) = 50$:

$$\frac{90}{1 + 271e^{-0.122x}} = 50$$

$$\frac{1 + 271e^{-0.122x}}{90} = \frac{1}{50}$$

$$1 + 271e^{-0.122x} = \frac{9}{5}$$

$$271e^{-0.122x} = \frac{9}{5} - 1$$

$$e^{-0.122x} = \frac{4}{5} \cdot \frac{1}{271}$$

now we take the natural logarithm on both sides:

$$-0.122x = \ln\left(\frac{4}{1355}\right)$$

$$x = \frac{\ln(4/1355)}{-0.122} \approx \boxed{48}.$$

This means that the percentage of some coronary heart disease is 50% at approximately 48 years.

$\square$