Math 105

Fall 2009
Instructor: Professor Harms

Guided Notes

to Accompany Text:
Excursions in Contemporary Mathematics
(6th edition of Text by Tannenbaum)

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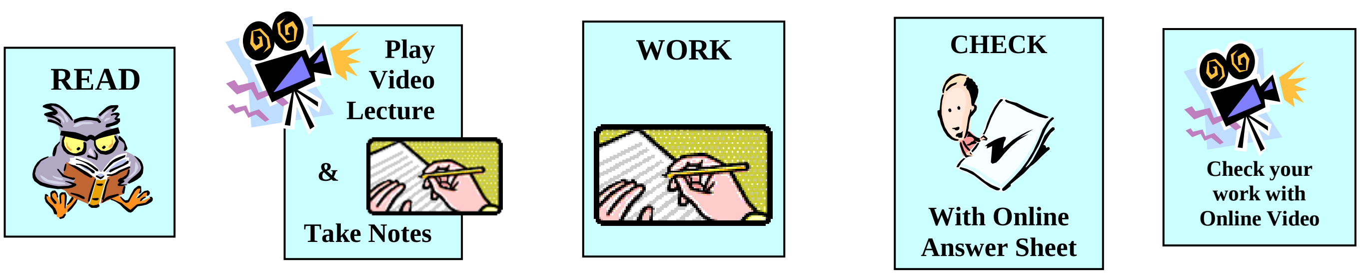
Part 1
Tutorial: Ratio, Proportion, and Percent

To the student:

- Part 1 is **<u>not</u>** associated with a chapter in your textbook. It is background material you will need in order to do the work in the text book.
- You must complete Part 1 before you can go on to Part 2.
- The rest of the guided notes (Parts 2 through 12) are closely associated with chapters in the text and have textbook homework assignments.
- **Watch for the following ICONS.**
  They give you directions on the guided notes pages:



Associated Online Resource
for Part 1
Guided Notes (GN)

GN Page	Type of Online Resource	Name of Online Resource
5	**Video Lecture**	Vertical and Horizontal Notation for Ratios
7	Answer Sheet	Practice Problem p. 7
8	**Video Lecture**	Proportion
9	Answer Sheet	More Practice on Ratio and Proportion p. 9
11	**Video Lecture**	Stacking Proportions Correctly
12	Answer Sheet	Practice Writing Proportion Equations Correctly p. 12
14	**Video Lecture**	More About Percents
15	Answer Sheet	Percent Practice p. 15
16	**Connect to Middle School**	Identify in Middle School Textbooks the presentation of Ratios, Proportions, and Percents, Relate to State Standards, and explain how to solve a proportion

READ

Tutorial: Ratio, Proportion, and Percent

Ratio, proportion, and percent are key ideas in many different types of mathematics problems.

While these are simple ideas, they are ones that are often only hazily understood by students coming into this course.

A clear understanding of the meaning and notation of ratios and proportions is necessary to successfully understanding many of the topics we will cover in this course including: relative frequency, probability, normal distributions, geometric growth sequences, the mathematics of voting and fair division.

So we will begin with an introduction to ratio and proportion at their most basic meanings and representations.

As you work through this unit, be patient and be thorough. Ideas that appear to be "obvious" when doing examples often seem much more difficult on quizzes and exams.

After going through this section, most students do very well working with ratio, proportion, and percent in the subsequent sections of this course.

So again, be patient and go through this material carefully, even if it seems obvious to you.

READ

Ratio

A **ratio** is a statement of a consistent relationship between quantities.

Ratios can be indicated by several different word patterns.
Here are some patterns you are responsible for recognizing as ratio word patterns.

Pattern: "__________ for every ____________."

Example: There are 2 boys for every 3 girls.

Pattern: "__________ to every _____________."

Example: The recipe requires 2 cups of sugar to every 5 cups of flour.

Pattern: " __________ per ____________."

Example: The car travels 65 miles per hour.

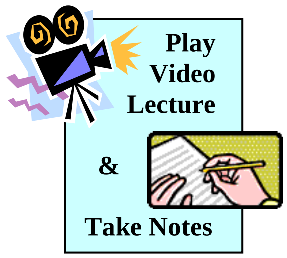
Example: The dosage is 12.5 mg per 25 pounds of body weight.

Pattern: "_____ to ______."

Example: "The ratio of wins to losses is 3 to 1".

Note: **PERCENT is a special type of ratio that compares to 100**.

15% (15 per "cent") can be interpreted as the ratio $\frac{15}{100}$



Vertical & Horizontal Notation for Ratios

Ratio phrase: 2 boys for every 3 girls

Horizontal Notation:

Vertical Notation

Remember that in writing ratios the order matters.

If the ratio of red to green is 4 to 5,

Horizontal Notation

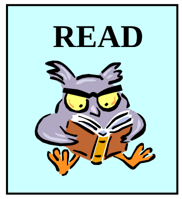
(a) green to red

(b) red to green

Vertical Notation

(a) green to red

(b) red to green



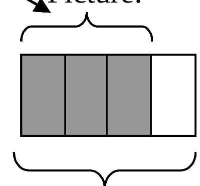
The Difference Between a Ratio and a Fraction

A **fraction** always compares

a **<u>number of parts</u>** to the **<u>total number of equal-sized pieces</u>.**

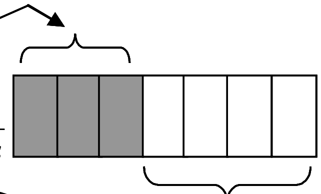
Picture:

For instance: $\frac{3}{4}$ as a fraction means $\frac{3\ \textit{shaded pieces}}{4\ \textit{total pieces}}$



A **ratio** compares a **<u>number of parts of one type</u>** to a **<u>number of parts of another type</u>**. The **<u>type of parts</u>** is critical to the correct interpretation of the ratio.

the picture for $\frac{3}{4}$ as the ratio $\frac{3\ \textit{shaded pieces}}{4\ \textit{unshaded pieces}}$



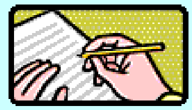
BUT The **<u>fraction</u>** of the rectangle above that is shaded is $\frac{3}{7}$

(7 is the **total number of equal-sized pieces**)

Note: Ratios can be reduced (simplified) just like you reduce fractions.

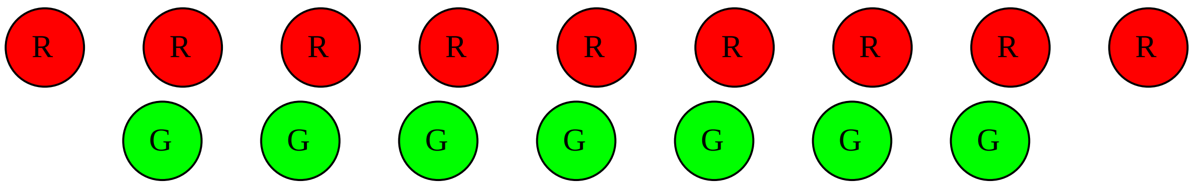
Both the fraction and the ratio $\frac{6}{8}$ can be simplified (reduced) to the notation $\frac{3}{4}$.

WORK



Practice Problem

1. Consider this Picture:



(a) What is the ratio of red dots to green dots?

(a) What fraction of the dots are red?

(a) What is the ratio of green dots to red dots?

(a) What fraction of the dots are green?

(a) What is the ratio of red dots to total dots?

2. There are 45 students in the 5th grade at Hawley Elementary School. The ratio of girls to boys in the 5th grade there is 4:5.

(a) What fraction of the students are boys?

(b) How **<u>many</u>** of the students are boys?

CHECK

with ONLINE
Answer Sheet

Proportion

A **proportion** is a statement of to two different ratios that name the same basic relationship.

Consider the picture examples below.

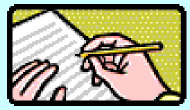
Notice that in each case the **relationship** remains the same:
there are 3 O's for every 2 X's.

Case 1: OOO
XX
A ratio of _______ to _________.

Case 2: OOO OOO
XX XX
A ratio of ________to _________.

The proportion demonstrated with the two cases above is written:

WORK



More Practice on Ratio and Proportion

Complete the following problems. The ratio of O's to X's remains the same throughout this exercise.

Case 3: OOO OOO OOO A ratio of ________ to _________.
XX XX XX

Case 4: If there are 21 O's

There should be _______ X's?

Case 5: There should be ________ O's

When there are 32 X's.

Case 6: 40% of the symbols are X's.

Write this as a properly stacked proportion

CHECK

with Online Answer Sheet

READ

Setting up a Proportion Equation

The key to setting up proportion equations correctly is that you must “stack” the ratios in a proportion correctly. Corresponding parts of the ratios must be in corresponding positions in the equation.

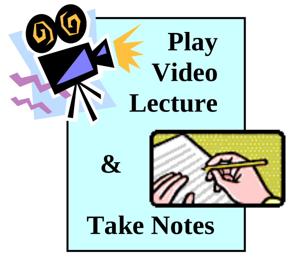
Then to create an equation we must remember that to be proportional, the two ratios must be equal.

This can be thought of as filling in a table like the one below for the problem “the ratio of boys to girls in the 4th grade class is 2:3. There are 200 boys and ***x*** girls in the school.”

Source→ ↓ gender	4th grade	Whole School
Boys		
Girls		

Proportion Equation

Go on the next page to see this worked out



Stacking Proportions Correctly

"The ratio of boys to girls in the 4th grade class is 2:3."

Write the proportion equation if the same proportion holds true throughout the whole school and there are 200 boys and ***x*** girls in the school."

Source → / ↓ gender	4^{th} grade	Whole School
Boys		
Girls		

→

Proportion Equation

Notice that the same problem could also be correctly "stacked" this way:

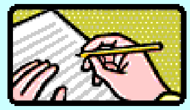
Gender → / ↓ Source	Boys	Girls
4^{th} grade		
Whole School		

→

Proportion Equation

Generalization: Any "stacking" that keeps the categories line up in corresponding positions is a correct way to write the proportion.

WORK



Practice Writing Proportion Equations Correctly

Write two different, correctly stacked proportion equations for this proportion situation:

The number of representatives of each party is proportional to the number of voters of each party. There are 54 democratic representatives, 76 republican representatives, 7980 republican voters, and 5670 democratic voters.
(BE CAREFUL!! – think of it as a table)

When you are finished :

Check your work with Online Video

READ

Other Properties of Proportions

Proportions can also sometimes be solved by using the properties you are familiar with when finding equivalent fractions.

You can **multiply both the top and bottom** of the ratio by the same amount to produce a ratio that is equal to the ratio you started with.

× 289

$$\frac{1}{2} = \frac{x}{598}$$

× 289

1 × 289 must be equal to ***x***

BECAUSE

2 × 289 is equal to 578

You can also **divide both the top and bottom** of the ratio by the same amount to produce a ratio that is equal to the ratio you started with.

÷ 90

$$\frac{180}{270} = \frac{2}{x}$$

÷ 90

180 ÷ 90 is equal to 2

SO THEREFORE

270 ÷ 90 must be equal to ***x***

More About Percents

A. Any type of percent problem can be done by setting up a proportion and solving it using the cross-multiply and divide method. The basic percent problem proportion has the following form:

$$\boxed{\frac{\%}{100} = \frac{is}{of}}$$

Example: 12 is what percent of 60?

Example: 40% of what is 128?

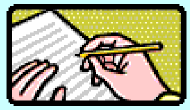
B. When you are required to find a "percent of a number" you can use this short-cut:

Step 1: rewrite the percent as a decimal.
You can do this by dividing the amount in front of the percent sign by 100.

Step 2: multiply this decimal times the number you are taking the percent of

Example: 4.5% of 98

WORK



Percent Practice

1. Rewrite as an equivalent decimal.

(a) $\frac{94}{1000}$ (b) 8% (c) 0.0007% (d) 150%

______________ ________________ ______________________ ___________________

2. Rewrite as the equivalent percent.

(a) $\frac{3}{5}$ (b) 0.65 (c) 1.2 (d) .0000025

_______________ ________________ _____________________ ____________________

3. Write the proportion and find the required quantity.

a. 25% of 3450 is ________________.

b. What percent is 40 of 250? __________

c. 75% of __________ is 150.

d. 398 is 20% of _______________.

e. Find 35% of 68.

4. Use the shortcut method to find 3.5% of 40. _____________

CHECK

With Online Answer Sheet

End of Chapter Checklist

Before beginning the next Part be sure you have done the following:

______1. Worked through all the guided notes
for Part 1: Ratio, Proportion, & Percent

______2. Worked through all the accompanying online video sessions
for Part 1: Ratio, Proportion, & Percent

______ 3. Checked all the "WORK" pages with the online answer sheets
for Part 1: Ratio, Proportion, & Percent

______4. Middle School Connections

B) Include photo copies of examples of each of the following: Ratios, Proportions, & Percents presented in middle grade - gr 6-8 math book(s), list sources that have book name, authors, publisher, and copy right date.

C) List the MN 2007 Math Standards http://education.state.mn.us/mde/Academic_Excellence/Academic_Standards/Mathematics/index.html
or ND Math Standards http://www.dpi.state.nd.us/standard/content/math/index.shtm that describe the standards or benchmarks that include Ratios, Proportions, & Percents

C) Be prepared to explain how to solve a proportional problem in at least 2 different ways.

Part 2: Collecting Statistical Data

To the student:

- This Part corresponds to Chapters 13 in the textbook
- First, read Chapter 13. Read for the BIG IDEAS and basic meaning. Don't worry about the details. We will do that in the guided notes.
- Then work through these guided notes
- Then do the homework assignment below
- Homework: Pages 467-473 #1-3, 9-12, 9, 21-24, 29, 30, 39-42, 60, 63

Associated Online Resource for Part 2 of Guided Notes (GN)		
GN Page	**Type of Online Resource**	**Name of Online Resource**
20	Answer Sheet	Identifying Populations
21	**Video Lecture**	Computing a Population Characteristic
22	Answer Sheet	Problem 2
23	**Video Answer**	A Problem with a Twist
24	Answer Sheet	Case Study #1 The U.S. Census
26	**Get to Instructor by Sept. 3rd**	Case Study #2 The Literary Digest Poll
27	Answer Sheet	Survey Methodology
28	Answer Sheet	Quota Sampling
30	Answer Sheet	Case Study #3 The 1948 Presidential Election
32	Answer Sheet	Case Study #4 Modern Public Opinion Polls
34	Answer Sheet	Statistic/Parameter
36	Answer Sheet	Sampling Rate and Sampling Error
38	**Video Lecture**	Capture-Recapture Methodology
39	Answer Sheet	Capture-Recapture Problem
43	**Video Lecture**	Association is NOT Causation
45	Answer Sheet	Clinical Studies #1
46-47	Answer Sheet	Clinical Studies #2
48	Answer Sheet	Case Study #5 The Alar Case
49	**Get to Instructor by Sept. 9th**	Case Study #6 The Salk Polio Vaccine Trials
	Online Quiz #1 by	Part 2 Online Quiz

	Sept. 10th	

READ

Why Study About Statistics?

We live in the information age. We are bombarded with information everyday.

Most often that information is summarized in a numerical form like percentages. We use such information to make important decisions like:

- Which brand to buy
- Which candidate to vote for
- What foods to eat
- Which pharmaceuticals to use or medical treatments to undergo.

Therefore we need to become savvy consumers of information presented in these numerical forms.

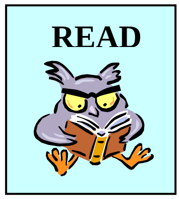
Information in numerical form is called **data.**

Statistics is the science of handling data including:

- Collecting data
- Organizing data
- Understanding and interpreting data

In order to make informed decisions based on information in statistical form, we need to be able to discern the meaning and trustworthiness of the statistics.

In this unit we will learn about methods for collecting data. Some collection methods produce trustworthy data. Other collection methods produce data that is inherently flawed and misleading. We will study how to determine the trustworthiness of the ways in which the data in a statistical study is collected.



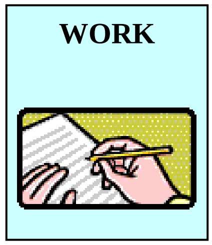
Part 13.1 in Text: The Population

Population: the entire group we want to describe with the statistics; the individuals or objects the statistics refer to. The population may be people, animals, or things. Usually we are unable to get information about the entire population. Usually we deduce or infer the information about the population based on a statistical sample. Often the population we are trying to make inferences about is called the "target" population
.

Sample: a selected part of the population; a subgroup of the population

N-Value: Capital "N" in statistics stands for the size of the actual population. The size of the population may change with time, so N-values may also change with time. Therefore it is important to include a time-frame when defining the target-population.

***n*-value**: Small "n" in statistics stands for the size of the sample.



Identifying Populations, Samples, and Their Respective Sizes

In each case, identify the population, the population size "**N**", the sample, and the sample size "***n*".** (If one of these is unknown, write "unknown").

A. In a 2003 study of the drinking habits of college students in the United States, we asked 500 college students at MSUM their age and how many alcoholic drinks (cans, bottles, or glasses) they consume each week.

Population:____

Sample: ____

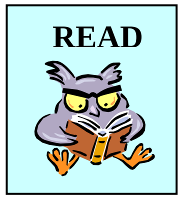
N-Value: _____________ Sample size _______________

B. A polling company working for a presidential candidate during a state primary surveys a sample of 3500 registered voters in the state of New York on November 1st 2004 to determine if they are in agreement with the candidate's stand on abortion.

Population:____

Sample: ____

N-Value: ___________ Sample size: _________________



Computing a Population Characteristic from a Representative Sample

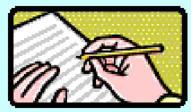
Representative Sample: a subgroup of the population that proportionally represents the qualities of the population. (Notice the idea of proportionality is important, here.)

IF the sample is proportionally representative of the population in terms of the characteristic being examined, **THEN** the following equation will be a true proportion (that is, the two ratios involved will be equal.)

$$\frac{\#\ \textit{with characteristic in the sample}}{\textit{total number in the sample}} = \frac{\#\ \textit{with characteristic in the population}}{\textit{total number in the population}}$$

Example: A sample of the population of MSUM students shows that 15% are Native American. If this sample is representative of the population of MSUM students, and there are 8,000 students enrolled at MSUM that semester, how many are Native American?

WORK

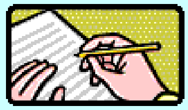


Problem 2: A fishery has 2,500 newly hatched fish in one of its nurseries. A sample of 50 is taken and examined for a threatening type of fin fungus infection. Of those 50, 7 show symptoms of the fungus infection. IF the sample is representative of the population with respect to the fungus infection, how many of the fish in this nursery are infected?

CHECK

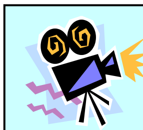
With Online Answer Sheet

WORK



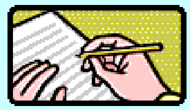
A Problem with a Twist

Problem: 50,000 fans buy tickets for the Great American Hard Rock Tour. Only 250 of those fans are over the age of 50. At one stop on the tour, there are 4500 seats in the auditorium and the house is sold out. IF this stop on the tour is representative of the total tour with respect to the age distribution of the audience, how many in the audience are **under the age of 50**?
(Be careful)



Check your work with Online Video

WORK



Case Study #1   Page 451 of Text

Directions: Read Case Study #1 on p. 451 of your text. Then answer the following questions. You will be turning this page in to your instructor. The point values for each question indicate the number of details expected for a full-credit answer.

1. Describe clearly the population the U.S. census is supposed to count.

2. Describe two historical complications that have occurred in defining this population.

3. Describe the purposes for which the statistical data of the U.S. Census is used.

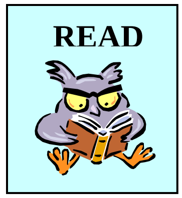
4. Why is it difficult to accurately count the resident population of the U.S.?

5. Explain what the term "differential undercount" refers to in terms of the U.S. census and explain why migrant workers, ethnic minorities, and urban poor are consistently undercounted more often than other groups.

6. The U.S. Census cannot be conducted using the sampling methods described in this section. Why not?

CHECK

With Online Answer Sheet



Samples: Representative or Biased?

Census: getting the information about every single member of the population. An example would be the U.S. Population Census. Sampling, therefore, is not used in a census.

.

Survey: a research method which collects data only from a selected subgroup of the population and then use these data to draw conclusions and make statistical inferences about the entire population. To correctly predict characteristics of the population from a sample, the sample must be representative of population.

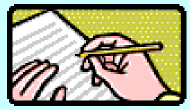
Selection Bias: A tendency built into the sampling methodology (whether intentionally or not) that systematically under-represents a group or characteristic within the population.

Sampling frame: the actual source from which the sample is selected. For instance, if the sample is selected from the phone book, the phone book is the sampling frame. In such cases, the population is usually intended to be the entire population of the geographic area covered by the phonebook. But in reality, not everyone in the area has a phone or allows their number to be listed, so not everyone in the target population is in the sampling frame.

Poor selection of the sampling frame can seriously compromise how well the sample represents the target population.

Sampling error: How much the sample prediction missed the actual population value. For instance, the difference between the percent of votes for a candidate predicted by a survey and the actual number of votes the candidate gets in the election.

WORK



Case Study #2 The 1936 Literary Digest Poll

Directions: Read Case Study 2 on page 453 of your text book. Then answer the following questions. You will turn in your answers to this page to your instructor by email or drop it off by Sept 3rd.

1. Complete the table below to compare the Literary Digest Poll predictions for the two candidates with the actual voting percentages. (2 points)

Candidate	**Poll Prediction**	**Actual Voting Outcome**
Landon		
Roosevelt		

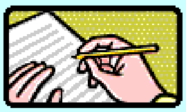
2. Sampling error is the difference between the percent of the vote the survey predicts the candidate will receive and the actual percent of the vote the candidate receives. Give the amount of the sampling error in the 1936 Literary Digest Poll and show how you computed this. (2 points)

(b) In previous elections, the Literary Digest Poll had predicted the election outcome correctly. What factors in the 1936 Literary Digest Poll contributed to its 1936 prediction of by such a large amount. (2 points)

4. According to your text book, what are the Lessons of the 1936 Literary Poll? (2 points)

5. Explain how the 1936 Literary Poll exemplifies these two lessons. (2 points)

WORK



Directions: Each survey method described below is flawed. For each one, describe (a) the kinds of bias that may result from the survey method described and (b) predict how you think the survey results may be different from the true opinion of the stated population.

1. Survey families of violent crime victims to determine the attitude of Americans towards capital punishment.

(a)

(b)

2. Survey a random sample of 100 women from a membership list of a local businesswomen's club to determine the opinions of employed adult women about government funding for day care.

(a)

(b)

3. 350 of the 375 letters received about a gun control bill by a member of Congress are opposed to the bill.

(a)

(b)

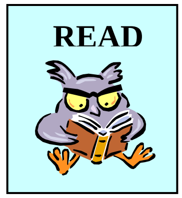
4. A television talk show conducts the following survey by asking their viewing audience to call one 800 number to vote "yes" and another 800 number to vote "no" on the following question: "If you had a terminal illness, would you want the right to end your life with a doctor's help?"

(a)

(b)

CHECK

With Online Answer Sheet

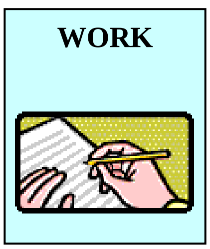


Quota Sampling

Quota sampling: systematic effort to force the sample to fit a certain national profile by using quotas. The proportion should be the same as those in the electorate at large, for example, selecting sample to represent a given percentage of:

- Men — Women
- Blacks — Whites
- Urban — Rural
- Age groups

The assumption of quota sampling is that every possible important characteristic of the population is accounted for by the quotas.



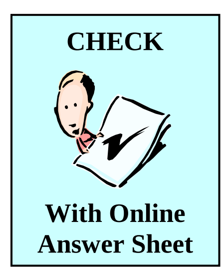
Suppose you are conducting a voter survey in which you want to use quota sampling with the following quotas:

40% white males over the age of 18
40% white females over the age of 18
6% black males over the age of 18
6% black females over the age of 18
8% randomly selected from the general population over age 18.

1. If you need 1600 people in your survey how many "white males over 18" should be selected for the sample?

2. If you need 1600 people in your survey how many "black males over 18 " should be selected for the sample?

3. Name a group in the U.S. electorate you would expect to be severely under-represented by this quota sample and explain why you would expect that group to be under-represented by this sample.



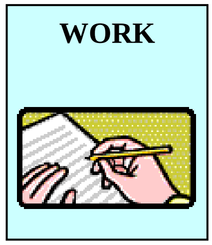
READ

Important Principles to Remember

- Always identify the population to which the statistical statement refers.

- A small sample can be representative of a large population **IF** the population is highly homogeneous (uniform, similar in characteristics). For example, only a small blood sample is required to check a person's blood for many characteristics because the characteristics are fairly uniformly distributed in the bloodstream at any given time.

- The more heterogeneous (the more diversity) in a population, the more difficult to find a representative sample. It is very difficult to get a representative sample of the population of the United States because of there is so much diversity.



Case Study #3 The 1948 Presidential Election

Directions: Read Case Study 3 on page 455 – 456 of your text book. Then answer the following questions.

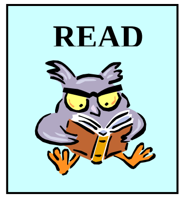
1. Complete the table below

Candidate	Predicted by Poll	Actual Election Vote
Truman		
Dewey		
3rd Party Candidate		

2. Name and describe the sampling method used in this poll?

3. According to your textbook, what are the lessons learned from this case study?

4. According to your textbook, how and why is quota sampling intrinsically flawed?



Section 13.3 Random Sampling

Random Sample: some form of chance decides who is selected for the sample. In theory, it is possible to get a random sample that is very biased – but it is highly unlikely when the sample used is large enough.

Simple random sample: any subgroup of the population should have the same chance of being selected for the sample as any other group of the same size.

Example: 20 names are put in a hat and mixed up. Two names are drawn at random. Any pair of the original 20 names has the same chance of being selected as any other pair of names.

Random samples are usually chosen by computer using a method that randomly selects the necessary number of names or items from the list that is entered using a "random number generator".

Data collected by random sampling has been shown both by practical experience and using mathematical theory to be reliable (trustworthy).

Random Samples are used in most:

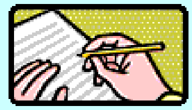
- Public opinion polls
- Industrial quality control

Difficulties with Random Sampling

- Random sampling is often not possible because it requires we have a list of the entire population to choose from frequently it is impossible to produce such a list. For example, a list of the entire population of the United States.

- Random sampling is often expensive and difficulty, because one an individual (person or item) is selected, it must then be located and surveyed. In large populations this can be cost and time prohibitive

WORK



Case Study #4 Modern Public Opinion Polls

Directions: Read Case Study 4 on pages 458-459 in your textbook. Then answer the following questions.

1. List two reasons why stratified sampling is a useful sampling method.

2. Strata are usually divided using what two types of categories?

3. Describe how a stratified sample of the population of the United States would be chosen, according to your textbook. Be thorough. Continue your answer on the back of this page if necessary.

4. When is a stratified sample a good choice (reliable, trustworthy) for gathering survey data?

5. List the steps taken by a Gallup Poll (p. 458) to create a stratified sample of the United States. Be thorough. Continue your answer on the back of this page if necessary.

6. Specifically describe how surveying using a stratified sample such as described for the Gallup Poll (p. 458) is more efficient in terms of time and cost than a simple random sample.

7. Paraphrase George Gallup's explanation about why the size of the sample does not have to be proportional to the size of the population (p. 459)

CHECK

With Online Answer Sheet

READ

Section 13.4 Formal Terminology and Key Concepts

Census: gather the pertinent data from every individual in the population

Survey: use a subset of a population as a sample and generalize from the sample to draw conclusions about population as a whole.

Statistic (singular): a number fact **from the sample**.

Note that **a statistic is always an ESTIMATE** of the population value because it comes from a sample.

Statistics (with an "s" on the end) has two meanings:

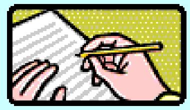
The first meaning is the one we have already studied: statistics is the science of data.

The second meaning is simply the plural of the word "statistic" described above. In this sense statistics refer to two or more number facts derived from a sample. You can see that the two ideas are closely related.

Parameter: a numerical value derived directly from the population, not from a sample. For instance, the capital N-value is the number of members of the population. Therefore N is a parameter.

Example: If you did a **census** asking the question, "Are you in favor of capital punishment?" you would be asking every person in the population and therefore the numbers you would get would be parameters because you got them from the population, not a sample.

WORK



Practice: Statistic or Parameter

4000 people from the Heartbreak, Nevada phone book are interviewed by phone about whether they consider themselves Republican, Democratic, or Independent. The official population of Heartbreak is 120,000. The interview results are

1,200 say they are Republican
2,300 say they are Democrats
500 say they are Independent

Using the information in the example above, circle the correct term from the underlined pair in each statement.

(a) 120,000 is a statistic/parameter in this example.

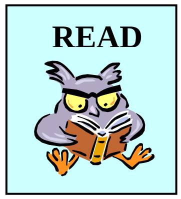
(b) 1,200 is a statistic/parameter this example.

(c) 2,300 and 500 statistics/parameters in this example.

(d) This is an example of a census/survey methodology.

CHECK

With Online Answer Sheet



Sampling: Error, Bias, & Variability

Sampling Variability: two different samples are likely to give two different sample statistics even when the samples are chosen using the same method. Sampling variability is unavoidable. With good sample selection and sample size, sampling variability error can be reduced, but can never be eliminated completely.

Sampling Bias: is the result of having a poorly selected sample.
It can be completely unintentional, because samples can be affected by many subtle factors. However, using proper methods of sample selection can eliminate sample bias.

Sampling Error: a number that describes the difference between a parameter value and the statistic used to estimate that parameter.

Sources of Sampling Error

- Chance error = sample variability
- Sampling bias

Sampling Rate: $\frac{n}{N}$ tells the proportion of the population that is being sampled.

- n represents the number in the sample
- N represents the number in the population
- Usually the sampling rate is reported as a percent.
- In a public opinion poll, a sample size of 1500 is sufficient to get reliable statistics. Remember George Gallup's soup analogy. Choosing a good sample is more important than having a very large sampling rate.

READ

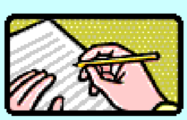
Example of Computing Sampling Error: A poll before the election says candidate Johansen will win with 54% of the vote. In the election, Johansen receives 57% of the vote.

The sampling error is found by subtracting the actual percent minus the predicted percent. The prediction was off by 3% so the sampling error is 3%. (57% minus 54%)

Example of Computing the Sampling Rate: From a population of 25 million, we use a sample size of 1250. Find the sampling rate as a percent.

$$\frac{n}{N} = \frac{\textit{number in the sample}}{\textit{total population}} = \frac{1250}{25{,}000{,}000} = 0.00005 = 0.005\%$$

WORK



Situation: A poll of 3000 people in a town of 36,000 finds that 1800 say they will vote for candidate Adams for mayor. In the election, 55% actually vote for candidate Adams.

1. What is the sampling rate for the poll in this situation?

2. What is the sampling error in this situation?

CHECK

With Online Answer Sheet

READ

Capture – Recapture Method of Sampling

Capture – Recapture is a common method used to estimate the size of a population by sampling. Biologists and ecologists use this method extensively to estimate wild animal populations.

> **Step 1**: **Capture** a sample of the animal you want to count in the area you want to know about. (your book calls the number captured)
>
> - **Tag** all the captured animals (given them each an identifying mark)
> - **Release** them back into the wild
>
> **Step 2**: **Recapture** after enough time for the released individuals to re-mix with the whole population, capture a new sample of individuals and count the number of tagged and the number of untagged individuals in this second sample.

The logic of capture-recapture computations: The computation is the proportion formed by two correctly stacked ratios.

<u>IF</u> we can assume that the recaptured sample is representative of the whole population, then

$$\frac{\#\ tagged\ in\ total\ \ population}{Total\ \ population} = \frac{\#tagged\ in\ the\ recapture\ sample}{Total\ \#in\ the\ recapture\ sample}$$

I suggest you memorize this rather than the formula on page 460 of your text.

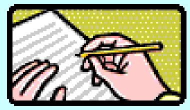
Notice that we KNOW the # tagged in the total population – that is the number we tagged in the first sample

Example 13.4 (reworded as such questions will appear on the exam)

A large pond is stocked with catfish. You capture 200 catfish, tag and release them. You wait enough time for the tagged fish to spread out and mix well with the general population. Then you capture another sample. This sample has 250 catfish. Of the 250 catfish in this second sample, 35 have tags.

If the second sample is representative of the catfish population in the pond, estimate the number of catfish in the pond.

WORK

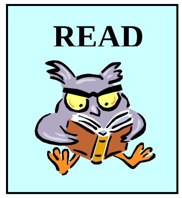


Capture – Recapture Sampling

Problem: In a rural county in Minnesota 60 deer are captured, tagged, and released. After 6 weeks, another sample, this time of 54 deer, is captured in the same area. Of those in the second capture, only 2 had tags. Estimate the number of deer in the area.

CHECK

With Online Answer Sheet



Section 13.5 Clinical Studies (Clinical Trials)

Clinical Studies do not collect data for the same purposes as surveys and censuses. Instead, Clinical Studies attempt to determine whether a single variable can cause a certain effect.

Experimental Variable:

Flu Vaccine

Causes??

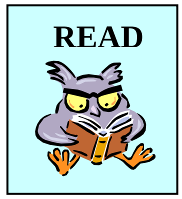
Effect:

Immunity to Flu Virus

New vaccines and drug treatments are put through clinical studies before being officially approved for public use.

Things that are “unhealthy” like cigarettes and caffeine are officially identified as “unhealthy” after clinical studies show that people who include significant amounts of them in their lifestyle have more health problems than people who do not include them.

Even clinical studies that are properly designed can lead to conflicting conclusions. But when clinical studies of the same variable, done in different labs by different groups, **consistently find the same conclusion**, the clinical study method is persuasive.



Controlled Clinical Study Methodology

Confounding Variable: a characteristic (not the one being studied) in which the control and treatment groups differ. Then you can't tell whether the effect was due to the characteristic being studied or due to this other characteristic or a combination of both.

Two common confounding variables are

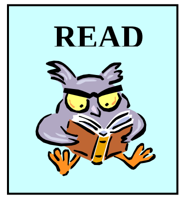
- **spontaneous improvement:** Often a patient's condition will improve without any treatment at all, or like acne it may improve naturally with maturation.

- **placebo effect**: Research shows that people receiving a **placebo** (a harmless, inactive substance like a "sugar pill") often report experiencing improvement. Apparently just the idea that one is getting treatment can produce positive results.

A **controlled clinical study** uses two groups:

- **treatment group** (receives the actual treatment)
- **control group** (sometimes called the comparison group) should only differ from the treatment group in that they do not receive the treatment.

Using a treatment and a control group (a controlled clinical study) is the best way to prevent the results from being confounded by spontaneous improvement, maturation, or the placebo effect.



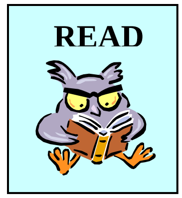
Blind Study: The placebo effect cannot be eliminated, but it can be controlled by giving a placebo to the control group and conducting a **blind study,** in which neither the treatment nor the control group know whether they are getting the real treatment or the placebo.

Double Blind Study: The scientists conducting the study are also not aware of whether the participant is getting the real treatment or the placebo. (participants and researchers are both "blind").

Even clinical studies that are properly designed can lead to conflicting conclusions. But when clinical studies of the same variable, done in different labs by different groups, **consistently find the same conclusion**, the clinical study method is persuasive.

Randomized Controlled Study: subjects are randomly assigned to either the treatment or control group

We can only deduce that the treatment CAUSES the effect <u>if the treatment group experiences the effect and the control group does not experience the effect</u>.

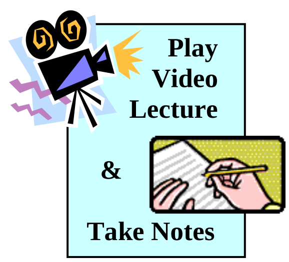


Association is NOT the same thing as Causation

Just because 2 conditions occur together **<u>does not mean</u>** one condition **<u>causes</u>** the other.

They may both be caused by some 3rd condition, or they may just coincide by chance.

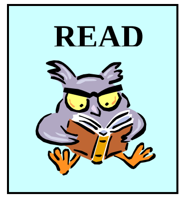
ALSO a single effect can have many possible and actual causes.



Example: The school district that receives the most federal money and pays the highest teachers' salaries is north of the artic circle. It has the lowest national test scores. Does higher teacher pay cause low test scores?

Example: A black cat crosses your path in the morning and by afternoon you have lost your job. Did the black cat crossing your path cause you to lose your job?

Example: In clinical trials in the 1950's, there was no indication that the drug thalidomide caused birth defects. Yet in the years immediately after it became available by prescription, thousands of cases of a particular type of birth defect occurred in newborns of women who were taking thalidomide.



Study #1 (fictional): In order to determine the effectiveness of a new drug for HIV treatment, the researchers conducted a study at the Park HIV Clinic in Philadelphia. The clinic first asked all 8,000 of their HIV patients who were between the ages of 20 and 40 years of age if they would be willing to participate.

Only 2000 volunteered to participate in the study. All 2000 of those volunteers were given a battery of medical assessments to determine the severity of symptoms they were experiencing and prognosis.

The researchers looked at the results of these medical assessments and found there were 150 of these volunteers who were in the beginning stages of HIV infection and were showing only minimal symptoms. These 150 patients became the participants in the study.

By random assignment, 75 were assigned to “Group A” and the other 75 were assigned to “Group B”. Group A received injections from “Drug A” vials while Group B received injections from “Drug B” vials.

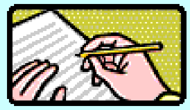
One vial was the experimental drug and the other vial was a placebo treatment.

Neither the patients nor the researchers knew whether “Drug A” or “Drug B” was the actual treatment drug.

Participants received the injections once a week for 6 months. At the end of the 6 months of treatment, the patients were again given the same battery of medical assessment to determine the severity of symptoms they were experiencing and prognosis. The average level of health was found to be significantly better for Group B.

Group B turned out to be the group that had received the real drug treatment.

WORK



Use the case study reading on the previous page to answer the following questions.

1. What is the sampling frame in this study?

2. What is the target population of this study?

3. Does it matter to the results of the study that the participants were volunteers? Why or why not?

4. What purpose did the initial medical screening to select the 150 actual participants serve in the methodology of this survey?

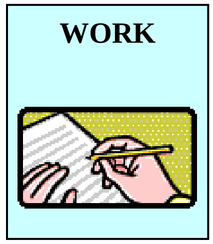
5. What makes this study a **controlled** study?

6. What makes this study a **randomized controlled** study?

CHECK

With Online Answer Sheet

7. Is this study **blind** or **double blind**? How can you tell?



Read the "case study" below and then answer the questions.

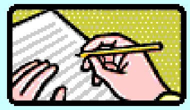
Study #2: In order to determine the effectiveness of a new vaccine that is alleged to cure "math anxiety", a clinical study was conducted. One thousand volunteer college students enrolled in math courses across the U.S. were chosen to participate in the study. The 1,000 students were broken up into two groups. Those enrolled in calculus courses or higher were given the real vaccine. The students in remedial and basic math courses were given a fake vaccine consisting of sugared water. None of the students knew whether they were being given the real or the fake vaccine, but the researcher conducting the experiment knew. At the end of the semester the students were given a test that measured their level of math anxiety. The students in the treatment group showed significantly lower levels of math anxiety than those in the control group. On the basis of this experiment the vaccine was advertised as being highly effective in fighting math anxiety.

1. The **sampling frame** in this study consists of
   A. the treatment and control groups
   B. all U.S. college students
   C. all U.S college students enrolled in math classes
   D. all students that suffer from "math anxiety"
   E. None of the above

2. The **target population** in this study consists of
   A. treatment and control groups
   B. all U.S. college students
   C. all U.S. college students enrolled in math classes
   D. all students that suffer from "math anxiety"
   E. None of the above

(continued next page)

WORK



3. The **control group** in this experiment consists of
   A. the 1,000 volunteer college students used for the study
   B. the students given the real vaccine
   C. the students given the fake vaccine
   D. This experiment has no control group because it used volunteers.
   E. None of the above

4. This experiment can best be described as a
   A. double blind randomized controlled experiment
   B. double blind controlled placebo experiment
   C. blind randomized controlled experiment
   D. blind controlled placebo experiment
   E. None of the above

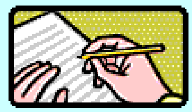
5. The results of this experiment should be considered unreliable because
   A. only college students were used
   B. the treatment and control groups were not the same size
   C. the sample was too small
   D. the treatment and control groups represented two very different segments of the population
   E. None of the above

6. Which of the following is most likely confounding variable for this experiment?
   A. the student's background in mathematics
   B. the student's grade level (freshman, sophomore, junior, senior)
   C. the type of college attended (two year, four year, university)
   D. the student's sex (male, female)
   E. None of the above

CHECK

With Online Answer Sheet

WORK



Case Study #5 The Alar Case

Directions: Read Case Study #5 in your text (pages 462-463). Then answer the following questions.

1. What is Alar?

2. Describe the experiment from which the conclusions were drawn.

3. Describe the conclusion that was announced in 1989 as a result of this study.

4. What effect did this have on the apple-growing industries and the state of Washington?

5. What are the methodological errors in this study?

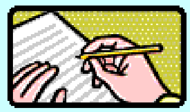
6. What did subsequent studies show?

7. "A single effect can have many possible and actual causes."
   Explain how this statement relates to the lesson learned from this case study.

CHECK

With Online Answer Sheet

WORK



Case Study #6 The Salk Polio Vaccine Field Trials

Instructions: Read Case Study #6 in your textbook on pages 463-465. Then answer the following questions. You will turn in this page to your instructor when you have completed it or type it up and email it to Professor Harms by Sept. 9th.

1. What are the differences between a live-virus vaccine and a killed-virus vaccine? (2 points)

2. Describe the vital statistics approach to testing a vaccine. (2 points)

3. The Salk trials were double blind. Explain what double blind means and why it was important in this case. (2 points)

4. In 1952 there were 60,000 reported polio cases.
   In 1953 there were 35,000 reported polio cases.
   Does this demonstrate the effectiveness of the polio vaccine? Why or why not? (2 points)

EMAIL YOUR ANSWERS TO THIS EXERCISE TO YOUR INSTRUCTOR

End of Chapter Checklist

Before beginning the next Part be sure you have done the following:

_____1. Worked through all the guided notes for
Part 2: **Collecting Statistical Data**

_____2. Worked through all the accompanying online video sessions
for Part 2: **Collecting Statistical Data**

_____ 3. Checked all the "WORK" pages with the online answer sheets
for Part 2: **Collecting Statistical Data**

_____ 4. **Turned in** your answers to Case Study #2 to your instructor by 9/3.

_____ 5. **Turned in** your answers to Case Study #6 to your instructor by 9/9.

_____6. Completed, checked and corrected all the homework problems from Chapter 13 in the text Pages 467-473 #1-3, 9-12, 9, 21-24, 29, 30, 39-42, 60, 63

_____7. Completed and turned in the online quiz for Part 2: Collecting Statistical Data by 9/10.

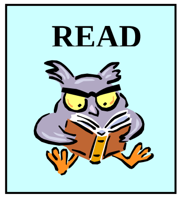
Part 3: Descriptive Statistics

To the student:

- This Part corresponds to Chapters 14 in the textbook
- First, read Chapter 14. Read for the BIG IDEAS and basic meaning. Don't worry about the details. We will do that in the guided notes.
- Then work through these guided notes
- Then do the homework assignment below
- Homework: Pages 500-506 #7-11, 13, 14, 21, 23, 33, 41, 46, 55, 56, 61, 65, 69

Associated Online Resource for Part 3 of the Guided Notes (GN)		
GN Page	**Type of Online Resource**	**Name of Online Resource**
53-54	**Video Lecture**	Organizing Data -- Tally
55	Answer Sheet	Create a Frequency Tally
56	Answer Sheet	Create a Frequency Table
57	Answer Sheet	Relative Frequency
58	Answer Sheet	Example: Relative Frequency
60	**Video Lecture**	Create a Bar Graph
61	**Video Lecture**	Misleading Graphical Representations
62-63	**Video Lecture**	Creating Circle Graphs
65	Answer Sheet	Categorical & Numeric Variables
67	**Video Lecture**	Histograms
69	Answer Sheet	Computing a Mean from a Frequency Table
71	Answer Sheet	Mean and Median Practice
72	Answer Sheet	Percentile
73	**Video Lecture**	Computing Percentiles
75	Answer Sheet	Practice with Quartiles
76	Answer Sheet	5-Number Summary
78	Answer Sheet	Box-Plot
79	Answer Sheet	Range
81	**Video Lecture**	Standard Deviation
	On-line Quiz #2 by Sept 17th	Online Quiz over Chapter 14

	Quiz in Class over Parts 1-3	This Quiz will be in class on Sept. 21st



Chapter 14: Descriptive Statistics

Descriptive Statistics: statistics that summarize or otherwise describe large amounts of numerical data.

- Present data visually as pictures or graphs
- Numerical summaries like **measures of center** and **measures of spread**

Data set: a collection of data values

Data points: the individual data values in the data set

Raw data: data as it was first gathered before any summarizing or computational manipulation

N is the size of the data set (the population of data)

Frequency: how often a particular data value occurs

Outliers: Extreme values in the data that do not fit the overall pattern of the data.

Tally: Organizing data by creating a table of all possible values and then making a mark in the table representing each data value.

Frequency Table: A table that lists all the data values and tells how often each data value occurs is the data set.

Relative Frequency Table: A table that lists all the data values and what percent of the data set occurred at each value.

Organizing Data

Data Set:

100	150	120	180	110	100	170	150	110	130
110	140	100	110	100	140	130	120	190	160
120	120	190	170	110	160	130	100	180	100
120	130	170	100	110	120	100	120	100	140
120	160	120	130	100	110	110	120	130	110

N = the number of data points in the data set =____________

Value	Tally
100	
110	
120	
130	
140	

Value	Tally
150	
160	
170	
180	
190	

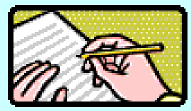
Continues on next page – same video lecture

(video lecture continues)

Frequency Table and Computing Relative Frequency

Value	Frequency	Fraction	Decimal	Relative Frequency
100	10			
110	9			
120	10			
130	6			
140	3			
150	2			
160	3			
170	3			
180	2			
190	2			

WORK



Create a Frequency Tally

Make a frequency tally for this **data set** of exam scores.

95, 90, 85, 90, 70, 15, 70, 50, 55, 80, 70, 80, 60, 45, 70, 75, 75, 75, 60, 65

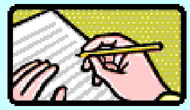
100	
95	
90	
85	
80	
75	
70	
65	
60	
55	

50	
45	
40	
35	
30	
25	
20	
15	
10	
5	

CHECK

With Online Answer Sheet

WORK



Create a Frequency Table

<table>
<tr><td>100</td><td></td></tr>
<tr><td>95</td><td>|</td></tr>
<tr><td>90</td><td>||</td></tr>
<tr><td>85</td><td>|</td></tr>
<tr><td>80</td><td>||</td></tr>
<tr><td>75</td><td>|||</td></tr>
<tr><td>70</td><td>||||</td></tr>
<tr><td>65</td><td>|</td></tr>
<tr><td>60</td><td>||</td></tr>
<tr><td>55</td><td>|</td></tr>
</table>

<table>
<tr><td>50</td><td>|</td></tr>
<tr><td>45</td><td>|</td></tr>
<tr><td>40</td><td></td></tr>
<tr><td>35</td><td></td></tr>
<tr><td>30</td><td></td></tr>
<tr><td>25</td><td></td></tr>
<tr><td>20</td><td></td></tr>
<tr><td>15</td><td>|</td></tr>
<tr><td>10</td><td></td></tr>
<tr><td>5</td><td></td></tr>
</table>

Translate the frequency tally above, into the frequency table below.

In a Frequency Table, you only include the scores that actually happened.

Use numerals rather than tally marks.

Scores	**95**	**90**	**85**	**80**	**75**	**70**	**65**	**60**	**55**	**50**	**45**	**15**
Frequency												

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Relative Frequency

Relative Frequency: the percent of the total population that had that value rather than the actual number that had that value.

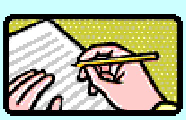
Relative frequency is used most commonly when the actual frequencies are very large numbers. This makes them easier to compare.

Steps for computing relative frequency.

1. Find N, the total number in the population. You can find this by adding up all the frequencies.

2. Then find the percents. Take each frequency and divide it by the total population. Convert the decimal to a percent by moving the decimal point 2 places to the right.

WORK



Finding the relative frequencies for these raw data.

Score	Frequency	Relative Frequency
5	3500	
4	2000	
3	1250	
2	1000	
1	250	

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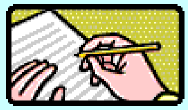

Example: In a high stakes exam used for academic scholarship awards, N=200,000 and the relative frequency of the of a perfect score is 0.04%.   How many students made a perfect score on the exam?

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Bar Graphs

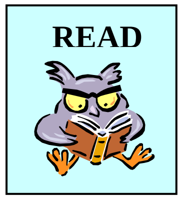
Bar graphs are often used to show frequencies. The higher the bar, the more frequent that data value.

Pictograph: uses pictures or icons to create the length of the bars

Characteristics of Bar Graphs:

- Bars are separated from each other, not touching each other.
- Height of bar indicates the frequencies of each score (or length of bar in horizontal)

- Bar graphs are usually limited to 12 or fewer bars. More than that is difficult to read.



Steps for Creating a Proper Bar Graph

Step 1: **Organize the data values**

Step 2: Make the vertical scale (usually) the frequencies. Use equal intervals and be sure to label the scale "frequency"

Step 3: Make the horizontal scale the possible values. Make it an equal interval scale, including values that did not occur. Include a word-label that tells what those values represent.

Step 4. Draw a bar above each value that did occur, making the bar as long as the frequency for that value. Keep the bars more narrow than the space between values so that the bars do not touch one another.

Important: If you are graphing **relative frequencies**, be careful to:

- Include the N-value in the title of the so that the actual data values can be reconstituted, if desired

- Be sure to label the frequency scale as "relative frequency" or "percent of total".

Create A Bar Graph

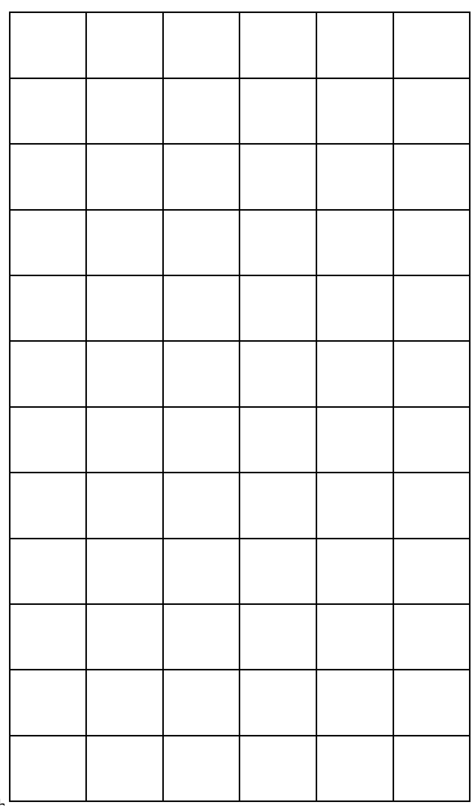
Create a bar graph to display this data set as <u>relative frequencies.</u> Be sure you label everything that needs to be labeled (see previous page).

The letters represent the letters of the correct answers on a multiple choice test.

A B A C B C C B C A A A C C B B B C B B C A
B A B C B B B C B A A B A B C C A A A C C B
B A A A B C

Step 1: Organize the Data and compute relative frequencies

Steps 2-4 Graph the values



Misleading Graphical Representations

Example from p. 553 of text: “Cheating” on the choice of starting value on vertical axis and stretching the scale on the vertical axis to make it look like there is more change than there actually is.

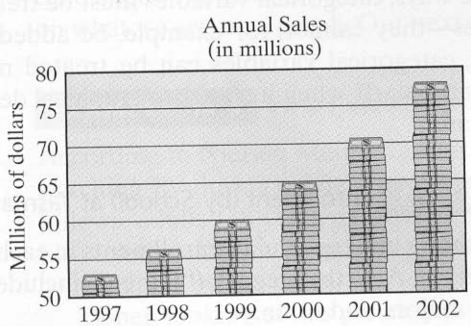


FIGURE 14-4
XYZ Corp. annual sales.

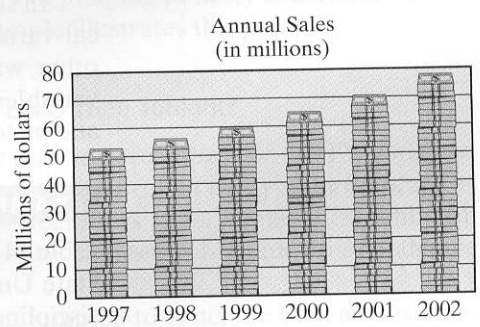


FIGURE 14-5
XYZ Corp. annual sales.

Example from P. 555 of text: When comparing characteristics of a population that is broken up into categories, it is essential to take into account the relative sizes of the various categories.

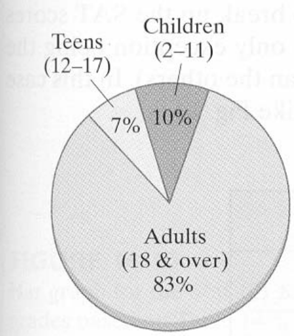


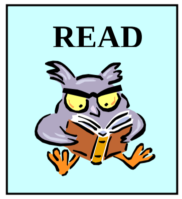
FIGURE 14-8
Audience composition for prime-time TV viewership by age group.
(*Source:* Nielsen Media Research.)

In the General Population:

Children (2-11 years) comprise 15% of the U.S. population.

Teens (12-17 years) comprise 8% of the U.S. population.

Adults comprise approximately 75% of the U.S. population.



Creating Circle Graphs (Pie Charts)

Circle Graphs (Pie Charts) are good for showing the respective sizes of categories within a whole population.

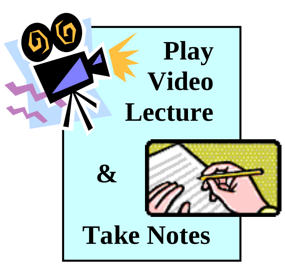
The circle represents the whole and the size of the sectors (the "slices") are proportional to the relative frequency of each category.

Remember 25% may be thought of as the ratio $\frac{25}{100}$.

Also remember that a circle contains 360˚. Half a circle contains 180 degrees. Each quarter of a circle contains 90 degrees.

It is often helpful to work with the reduced fraction for the percent rather than the number of degrees.

So the basic proportion relationship when figuring out the size of each sector is:



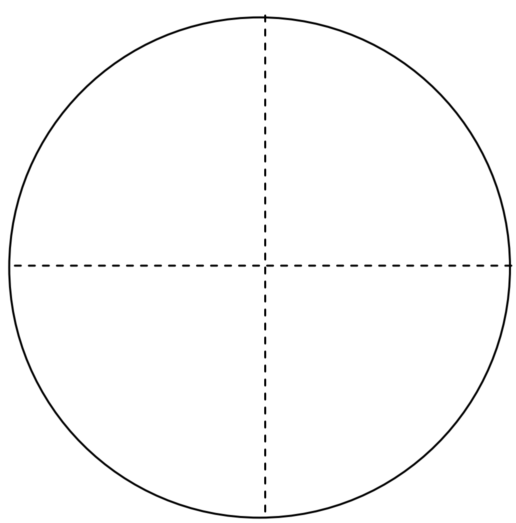
$$\frac{\%}{100} = \frac{\text{size of sector in degrees}}{360^\circ}$$

Find the size of the sector of a circle (in degrees) that would represent 25%.

(CONTINUES)

Make a circle graph (use a protractor) to represent the following data.

Type	Percent	Degrees
Red	25%	
Blue	50%	
Green	10%	
Yellow	10%	
Purple	5%	



READ

Section 14.2 Variables

Variable: in statistics a variable is any characteristic that varies within the population. Examples: what color, what size, what kind, how many of . . . These are all variables within a population.

Categorical Variable: (qualitative variable) represents a quality that is not normally measured numerically. For instance, gender. Categorical variables can be counted and quantified but we need to be very careful how we use those values and how we interpret them. We should not be giving a numerical average for variables like gender or hair color that are not actually numerical values to start with.

Numerical Variable: (Quantitative variable) represents a measurable quality.

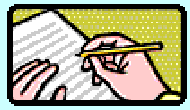
Discrete numerical variables are characteristics that cannot be measured in "infinitely" small fractions of a value (counting actual people, test scores, shoe sizes)

Continuous numerical variables are characteristics that can be measured in "infinitely" small fractions of a value (time, distance traveled, volume of liquid)

In the real world this distinction is blurred by

- **Rounding off** values that are actually continuous so they seem discrete (like measurements of length to the nearest quarter inch)
- **Calculations** or subdividing that create as many decimal places as necessary (often number of people, like 2.5 children per family is an "artifact" of a calculation such as taking the average)

WORK



Practice: For each situation below, indicate whether the variable should be considered **Categorical** or **Numerical**. If Numerical is discrete or continuous?

(a) Hair color: blond, brown, black, red, grey

(b) gender: male, female

(c) Shoe size: 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9, …

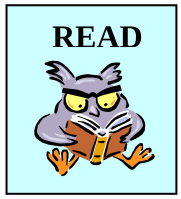
(d) Ethnicity: White, Black, Native American, Hispanic, Asian . .

(e) Height in inches

(f) Gender when asked to record it as a “1” if female and a “2” if male

CHECK

With Online Answer Sheet



Histograms and Class Intervals

Histogram: a variation of a bar graph showing relative frequencies.

Remember relative frequencies are the percent of the total population that had that value. In really large data sets where the raw values are very close together, we frequently group the raw data values into **equal-sized classes** and count how many raw data values fall in each of these **class intervals**.

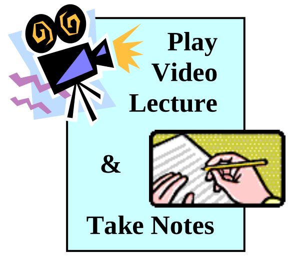
Important: When creating histograms, it is mathematically correct to draw bars for adjacent categories touching one another (different from regular bar graphs). This is because the class intervals are continuous, where as the original categories discrete. (letter grades like A, B, C, D, F are discrete, but class intervals like 91-100 are continuous).

Example:
GPA's for all students at MSUM would be an example of a large data set where the values are not well-separated.

The values only run from 0 to 4 and are computed to 3 decimal places so you get values like 3.725, 3.724, 3.726, 3.725 all of which are very close together and are pretty much the same GPA.

Teachers frequently make histograms of grades grouped by A, B, C, D, and F to look at the type of grade distribution in their classes.
This means that A+, A and A- are all recorded in the same bar.

This is another example of data where the values run together and can be categorized together.



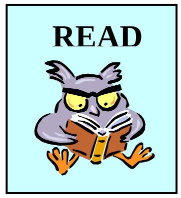
Histogram

Make a histogram of the following quiz averages. Use the final grade chart below to group the quiz averages into letter grades.

72	85.5	93.5	68	73.5
82.5	80	79.5	56.5	87.5
89.5	71	79.5	86	75
76.5	83	86.5	78	67

Grading Scale:

A	91-100
B	81-90.9
C	71-80.9
D	61-70.9
F	below 61



Section 14.3 Numerical Summaries of Data

Another way to summarize data and make large data sets more comprehensible, is to summarize them numerically. There are two main ingredients in such a summary:

<u>Measures of Location</u> (how the data "line-up" in an ordered list of the values)

Mean (Tells the **center** of the "weight" of the data)
Median (Tells the physical **center** of the data)
Percentile (Tells the **percentile-rank** the data value)
Quartile (Tells the **center AND the quarter-marks**)
"The Five-Number Summary"
(Minimum, Quartile 1, Median, Quartile 3, Max)
Box-Plots (Graphic representation of the 5-score summary)

<u>Measures of Spread</u> (how the data "bunch-up")

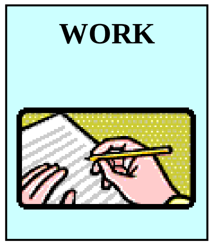
Range (Max – Min)
Interquartile Range (Q3 –Q1)
Variance (an intermediate step to get to the standard deviation)
Standard Deviation (average distance from the mean)

MEAN

Mean: The mean is the <u>arithmetic average</u>. The symbol for mean is the Greek letter Mu which looks like this : μ

It tells you where the center of the data is in terms of "weight" (balance point) or "volume" (equally full point -- If you think of the bars in the bar graph as being tubes filled with liquid, the average would be how full each tube would be if you used all the liquid, but completely evened-out the liquid so that each tube had the same amount in it.

The one disadvantage of the mean as a measure of center is that it is strongly influenced by extreme values (outliers).



Computing a Mean from a Frequency Table

To find the mean: Add up the scores and divide by the number of scores in the list. For instance, the mean of these scores: 50, 90, 75, 85, 70 is found by doing the following: $\frac{50+90+75+85+70}{5} = \frac{370}{5} = 74$

Be careful when you compute the mean for a frequency table of values.

Example: For the frequency chart below, you cannot get the average by adding just the numbers in the first row to get the total of the scores BECAUSE, for instance, the score of 9 did not happen just once, it happened 10 times.

score	1	6	7	8	9	10	11	12	13	14	15	16	24
frequency	1	1	2	6	10	16	13	9	8	5	2	1	1

If you need to compute the mean from a frequency table, it is easiest to add another row to the table that gives the "weighted" score.

score	1	6	7	8	9	10	11	12	13	14	15	16	24
frequency	1	1	2	6	10	16	13	9	8	5	2	1	1
Weighted score					**90**								

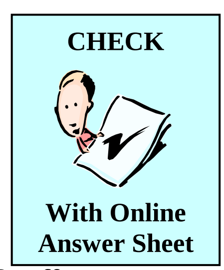
T

happened (the frequency). For instance, the score of 9 happened 10 times, so its weighted score is $9 \times 10 = 90$ because that is what you would get if you wrote out all 10 of them and added them up.

Fill in the rest of the weighted scores in the table above.

Then find the MEAN of ALL OF THE SCORES

$$= \frac{\text{Total of all the weighted scores}}{\text{Total of all the frequencies}} =$$



READ

Median

Median: The median the other measure of center that you are responsible for. It is the value which occurs in the middle of the data when the data are put <u>in numerical order</u>. So it is the center of the data, physically. Half of the data values are above the median and half are below it in this ordered list.

Computing the Median.
If there are an odd number of values in the ordered list, it is the center value. (For instance if there are 15 values in the ordered list, the median is the 8th value because it has 7 values above it and 7 values below it.)

If there are an even number of values in the ordered list, it is the <u>average of the center two values</u>. (For instance if there are 16 values in the ordered list, the median is the average of the 8th and 9th values in the list).

The Median is useful because it is a measure of center that is not influenced by extreme outlier values.

Consider this scenario: a company has a president and 4 other employees. Their salaries are:

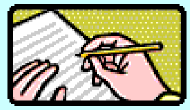
president:	\$250,000
employee 1:	\$20,000
employee 2:	\$17,000
employee 3:	\$17,000
employee 4:	\$16,000

The **mean** of the salaries is \$64,000. Would you like to work for a company where the average salary is \$64,000?

But the **median** of the salaries is \$17,000. Would you like to work for a company where the average salary is \$17,000?

In this scenario the median is a more representative measure of the center of the salaries
This demonstrates how an extreme value strongly affects the mean but leaves the median unaffected and why we use several different types of “averages”.

WORK



Mean and Median Practice

1. Data Set: 5, 10, 8, 7, 4, 10, 9, 5, 7, 8, 2, 1, 7, 6, 10

   (a) What is the mean of this data set?

   (b) What is the median of this data set?

2. Data Set: 100, 90, 70, 90, 20, 80

   (a) What is the mean of this data set?

   (b) What is the median of this data set?

3. Data Set:

Score	0	4	6	7	8	9	10
Frequency	2	1	2	1	5	8	6

   (a) What is the mean of this data set?

   (b) What is the median of this data set?

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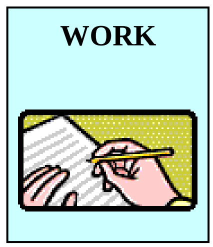
READ

Percentiles

Percentile: Percent and Percentile are NOT the same thing.

On a test, a score of **<u>90 percent</u>** means that you got, proportionally speaking, 90 out of 100 correct. It compares the amount you scored <u>to the total possible score</u>.

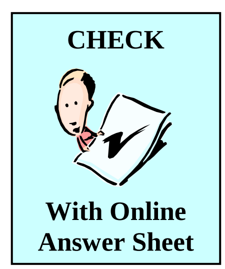
A percentile-rank compares how you did compared to everyone else. A **"90th percentile"** score means that, proportionally speaking, you did as well or better than 90% of the people who took the test. It compares your score <u>to all the other scores.</u>



Situation: Henry scored in the 87th percentile on the LSAT.

(a) Explain what "87th percentile" means.

(b) Did Henry score above average or below average? How can you tell?



READ

Computing a Percentile

Step 1: List the data in numerical order, from least to greatest.
Remember Percentile tells the relative position in the ordered list of data.
So think of the data as being an ordered list of values like

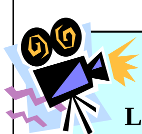
$d_1, d_2, d_3, d_4, \ldots, d_N$ where each of these have a numerical value, but they also have a position value (the subscript)

Step 2: Compute the locator for the pth –percentile using the formula:

$$L = \left(\frac{p}{100}\right)N$$

When L is a whole number: the $p^{th} - percentile = \dfrac{d_L + d_{L+1}}{2}$

When L is not a whole number: the $p^{th} - percentile = d_{L\ (rounded\ up)}$



Play Video Lecture

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Take Notes

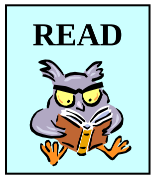
GPA Data Set

3.4, 3.9, 3.3, 3.6, 3.5, 3.4, 4.0, 3.7, 3.3, 3.8, 3.6, 3.9, 3.7, 3.4, 3.6

(a) Find the 80th percentile value for the following GPA's:

(b) Find the 75th percentile

(c) Find the 25th percentile



Quartiles

The Quartiles divide the data set into four quarters.

The data are first ordered.

Then find the median. Remember the median marks the middle of the data. 50% of the data is above the median and 50% of the data is below the median. Therefore the median is equal to the 50th percentile.

The locator for the 50th percentile is L=.50(13) = 6.5. Since this is not a whole number, round up to 7. The 7th term in the ordered list is 30 so the median = 50th percentile = 30.

10, 15, 20, 25, 25, 25, 30, 35, 40, 40, 40, 45, 50

Q1= d_4= 25

Q2 is 30

Q3= d_{10} = 40

Q1 (the first quartile) is the point below which $1/4$ of the data occur.

Therefore the first quartile is the same as the 25th percentile.

Example: Use the locator L= $\left(\frac{25}{100}\right)N$ to find the 1st quartile of the data set above.
L= .25(13) = 3.25

Since the locator is not a whole number, round it up to 4.

The 4th term in the ordered list is 25, so Q1 = 25

Q3 (the third quartile) is the point below which $\frac{3}{4}$ of the data occur.

Therefore the third quartile is the same as the 75th percentile.

Example: Use the locator L= $\left(\frac{75}{100}\right)N$ to find the 3rd quartile of the data set above.
L=.75(13) = 9.75

Since the locator is not a whole number, round it up to 10.

The 10th term in the ordered list is 40, so Q3 = 40.

Q4 (the fourth quartile is the maximum value since 100% of the data falls below that point.

WORK

Practice with Quartiles

Find the Median and Quartiles for this situation.

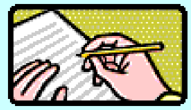
Example 14.14 (p. 563 of text)
During the last year, 11 homes sold in the Green Hills subdivision.
The selling prices, in chronological order, were:

$167,000
152,000
128,000
134,000
192,000
163,000
121,000
145,000
170,000
138,000
155,000

CHECK

With Online Answer Sheet

WORK



Five Number Summary

Five Number Summary (text p. 493)**: summarizes the data set by listing these five values, labeled and in order.**

Minimum
1st Quartile (Q1)
Median
3rd Quartile (Q3)
Maximum

Practice: Find the five-score summary for this data set:
7 4 10 8 5 6 4 6 1 3 7 5

CHECK

With Online Answer Sheet

READ

Box Plot (text p. 494)

Box Plot (sometimes called "box and whisker plots")

A box plot is a graphical representation of the 5-number summary.

Box Plots are good for comparing two similar data sets, for instance two different samples from the same population.

It gives a visual way to assess whether the two samples are significantly different or not.

Step 1: Draw an equal-interval scale that covers the entire range of values.

Step 2: Above the scale, draw a box that has Q1 as the location of one end and Q3 as the location of the other end.

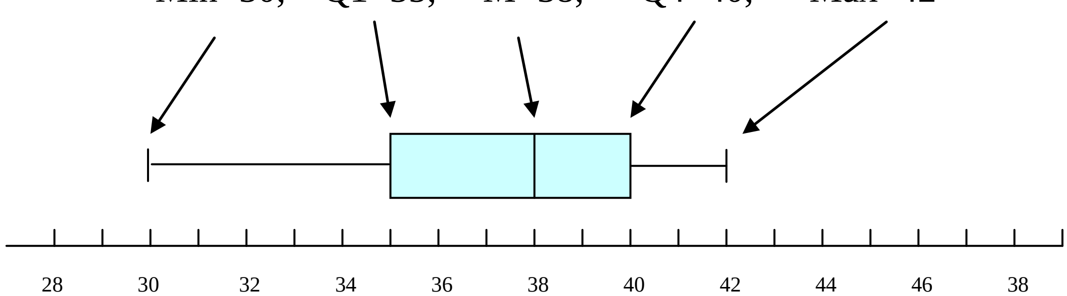
Step 3: Draw a line through the box indicating the Median, M.

Step 4. Locate the Minimum on the scale. Draw a "whisker" from the minimum to the nearest end of the box.

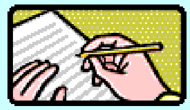
Step 5: Locate the Maximum data value on the scale. Draw a "whisker" from the maximum to the box.

Example: Box plot for

Min=30, Q1=35, M=38, Q4=40, Max=42



WORK



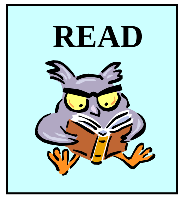
Data Set: 10, 11, 11, 12, 14, 15, 16, 16, 17, 18, 20, 25

(a) Find the 5-Number Summary

(b) Make a box-plot of these scores.

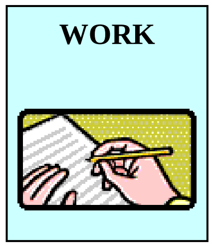
CHECK

With Online Answer Sheet



Range

Range (text p. 495) : tells how spread out the data are. To find the range, subtract the highest value minus the lowest value of the data set. Notice that the range depends only on the most extreme values of the data: the highest (maximum) and the lowest (minimum) values.



Practice:

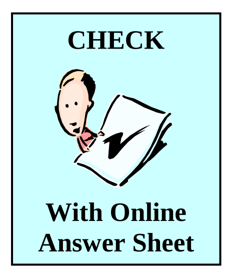
(a) Find the range of this data set:

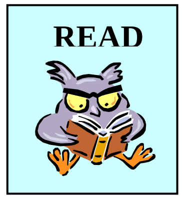
72 85.5 93.5 68 73.5 82.5 80 79.5 56.5 87.5

(b)Draw the box plot for the data in the data set above

(c)The **Interquartile Range is found by subtracting** Q3 -Q1.
Find the interquartile range for the data set above.

(d) What percent of all the data points must lie within the interquartile range? Why?





Standard Deviation

Standard Deviation (p. 496): the most important and most commonly used measure of spread for a data set. It represents the average amount that the data points differ (deviate) from the **Mean**.

Step 1: Find the average (Mean) of the data set.

Step 2: Make a table with three columns
and as many rows as there are data values.

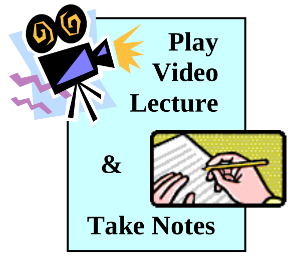
Step 3. In the first column, put the data values. It makes things easier
if you order them from least to greatest.

Step 4: In the second column put the answer to the computation
(Data Value -Mean) for each row.
Do all the subtractions in that order.
Some of these values will be negative.

Step 5: In the third column, square the value in the second column.

Step 6: Find the average of the values in the third column.
This average is called the "variance"

Step 7: The standard deviation, formally represented in statistics
by the small Greek letter "sigma" $\sigma = \sqrt{\text{variance}}$



Example: Find the Standard Deviation for this data set of test scores:
1, 6, 7, 8, 8, 9, 10, 11, 12, 13, 14, 15, 16, 24

Value	Deviation from Mean	Square of Deviation

End of Chapter Checklist

Before beginning the next Part be sure you have done the following:

_____1. Worked through all the guided notes
for Part 3: Descriptive Statistics

_____2. Worked through all the accompanying online video sessions
for Part 3: Descriptive Statistics

_____ 3. Checked all the "WORK" pages with the online answer sheets
for Part 3: Descriptive Statistics

_____4. Completed, checked, and corrected the textbook homework
Pages 500-506 #7-11, 13, 14, 21, 23, 33, 41, 46, 55, 56, 61, 65, 69

_____5. Taken the online practice quiz by Sept. 17th for Part 3: Chap 14-
Descriptive Statistics

_____6. Prepared for the first monitored **quiz** over Parts 1, 2, and 3 of the guided
notes by studying and online quizzes.

____7. Taken quiz over Parts 1, 2, and 3 on **Sept. 21st**.

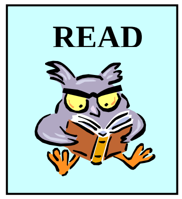
Part 4: Chances, Probabilities, and Odds
(Chapter 15 in textbook)

To the student:

- This Part corresponds to Chapters 15 in the textbook
- First, read Chapter 15. Read for the BIG IDEAS and basic meaning. Don't worry about the details. We will do that in the guided notes.
- Then work through these guided notes
- Then do the homework assignment below

Homework: Pages 531-536 #1, 3, 9, 11, 14, 15, 19, 30, 33, 34, 39, 59, 62, 69

Associated Online Resource for Part 4 Guided Notes (GN)		
GN Page	**Type of Online Resource**	**Name of Online Resource**
86	**Video Lecture**	Finding and Organizing Sample Spaces
87	Answer Sheet	Sample Space Practice
90	Answer Sheet	Practice Working with Factorials
92	Answer Sheet	Combination and Permutation Practice
93-94	**Video Lecture**	Notation and Formulas for Combinations & Permutat
95	Answer Sheet	Computing Combinations and Permutations
97	**Video Lecture**	Probability Assignments
98-99	**Video Lecture**	Finding "Events" (Target Sets)
100	Answer Sheet	Practice Finding Target Spaces
102-103	**Video Lecture**	Finding Probabilities Part 1
104	Answer Sheet	Practice Finding Probabilities
106-108	**Video Lecture**	Finding Probabilities Part 2
109	Answer Sheet	Multi-Stage Probability Practice
111	Answer Sheet	Odds and Probabilities Practice
112	**Chap 15 Online Quiz**	



Section 15.1   Chance, Probability, and Odds

Random experiment (probability experiment): an activity or process whose outcome cannot be predicted ahead of time.

> While the outcome of a single random event cannot be predicted ahead of time, there are patterns over the long run which can be used to measure the degree to which various outcomes are more or less likely. For instance, when you toss a coin, the outcome of head or tails is random and cannot be predicted ahead of time. But when a coin is tossed 100 or 1000 times or more, there is a pattern that the number of heads will be approximately equal to the number of tails.

Three ways to express the degree of uncertainty

- **Chance**: in mathematics, a measure of the degree of uncertainty, usually measured as a percentage.

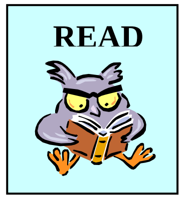
- **Probability**: in mathematics, a more formal measure of the degree of uncertainty. A probability must be a value between 0 and 1 (inclusive), with 0 being an impossible event and 1 being a certain event. Therefore, properly speaking, probabilities should be represented as a fraction or a decimal.

- **Odds**: in mathematics, another measure of the degree of uncertainty, properly expressed as a ratio.

Sample space: the set of all possible outcomes of a probability experiment. The symbol for the sample space is capital **S**.

When listing sets we use set brackets { } to surround the members of the set. The number of members in the set is called **N.**

The first step in computing mathematical probabilities is to identify and organize the sample space.



Set Theory for Probability

(not specifically in the book, but it will help you read the book)

In mathematics a **set** is a **well-defined** collection. It can be a collection anything: numbers, letters, people, animals, colors . . .
A set is defined in writing by using **set brackets** that look like this: { }

Anything inside the set brackets is a member or "element" of the set.

Sets are equal only when they contain exactly the same members. The **<u>do not</u>** have to be in the same order. For instance {1, 2, 3} is the same set as {2, 3, 1}.

A **subset** is any collection that can be made using only the members of the original (parent) set. So if set A = { H, T} is the original (parent) set, it has 4 subsets:

- { } the empty set. The empty set is a subset of every set. Symbol: $\varnothing$
- {H}
- {T}
- {H, T} the set is always a subset of itself.

Sample Spaces are sets of possible outcomes for a probability experiment. Since an **Event** is any subset of the Sample Space, it is useful to know how many subsets any set will have. That way you know how many possible Events exist for a Sample Space. The number of subsets for a set with N members in it is 2^N.

How many subsets will the set C = {blue, red, green, yellow, turquoise} have?

- There are 5 members in set C, so there will be $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$ subsets

When asked to **list all the Events for a Sample Space**,

- First figure out how many subsets there should be
- Then list them in a systematic way.
- Remember { } and the set itself are always subsets of any set. It is usually easiest to the subsets by the number of elements they have: start with the empty set (0 elements), then do all the subsets with 1 element, followed by all the subsets with 2 elements . . . up to the subset that is the entire set.

Finding and Organizing Sample Spaces

A. Toss a single fair coin once

B. Toss a coin three times
And record the order.

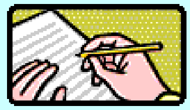
C. Toss two dice and
record the sum of the face values

D. Draw a marble from a bowl
with 3 red marbles, 2 green
marbles, a yellow marble,
and a blue marble.

D. Toss two dice, one red and one green, and record the outcome (red, green)

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)					
(3, 1)			(3, 4)		
(4, 1)					
(5, 1)					
(6, 1)				(6, 5)	

WORK



Sample Space Practice

Write out the sample space for each experiment below in set brackets { }.

1. Toss a coin twice and record whether it lands heads or tails on each toss. Record the order the outcomes occur.

2. Toss two coins at once and record how they land (order cannot be determined).

3. Toss a coin 4 times and record how many tails you get.

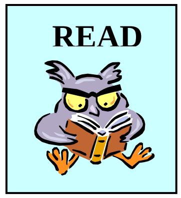
4. Shoot two free throws and record, in order, what happened (whether success ***s*** or failure ***f***).

5. Four candidates (candidate A, B, C, and D) are running in an election. The top three finishers are chosen President, Vice President, and Secretary in that order. Record the possible ways that (President, Vice President, Secretary) can be chosen.

6. Toss a pair of dice and record the product (multiplication answer) of the faces.

CHECK

With Online Answer Sheet



Counting: The Multiplication Rule

Sometimes I call this the "Buffet Principle". If you go to a buffet where for the buffet price you get to pick:

- 1 salad out of **4** salad choices
- 1 bread out of **3** bread choices
- 1 main dish out of **2** main dishes choices
- 1 vegetable out of **4** vegetable choices
- 1 desert out of **5** desert choices

The total number of ways to choose a complete meal on this buffet =
$4 \times 3 \times 2 \times 4 \times 5 =$ 480 different complete meals possible

Whenever you have a sequence of choices (or probability experiments), and you want to know how may possible outcomes are in the sample space, you can find the answer by multiplying the number of ways each of the individual choices (or experiments) can happen. Since finding N = the number of members of the sample space is a first step in computing probability, this is a very helpful principle to remember and use.

Example 15.2 (page 512 in textbook) Suppose we toss a coin three times and observe on each toss whether it lands heads or tails, how many outcomes should be in the sample space?

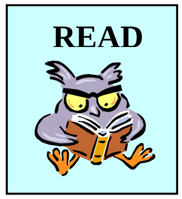
Compute number in sample space: ______________________

Write out entire sample space: ____

Example 15.10 (page 516 in textbook) Imagine that you want to buy a single scoop of ice cream. There are two types of cones available (sugar cones and regular cones) and three flavors of ice cream to choose from (strawberry, chocolate chip, and chocolate).

Compute number in sample space: _________________________

Write out entire sample space: ____________________________________



Drawing Without Replacement (Factorial!)

When the number of ways a thing can be done is reduced by 1 in each step, the resulting number of total ways the thing can be done is n!. The exclamation point is the "factorial" sign in mathematics and "n!" is read as "n-factorial", where n is the amount you started with.

Examples: $4! = 4\times 3\times 2\times 1 = 24$

$5!\ 5\times 4\times 3\times 2\times 1 = 120$

Suppose that the numbers 1-8 are written on slips of paper and put in a hat. If we randomly draw out each slip, and record the values on the slips, in order, without replacing them, then the number of ways that the 8 values can be drawn is:

$$8! = 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1 = 40{,}320 \text{ ways.}$$

Notice that these values get large very quickly, so the notation n! is a very useful shortcut for expressing these values.

Special Facts about Factorial:

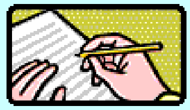
0! = 1. There are several reasons for this. If we think of this as "how many ways can you order the boys and girls in a family with 0 children?" there IS **1 way**: zero boys and zero girls. Remember we are counting ways to order the genders, not the number of either gender.

1! = 1 This seems reasonable.

You can't add or multiply factorials. For instance, 3! + 4! **is not** equal to 7!

You can simplify ratios of factorials by thinking about the multiplication involved: $\frac{5!}{3!} = \frac{5\times 4\times 3\times 2\times 1}{3\times 2\times 1} = \frac{5\times 4\times 3!}{3!} = 5\times 4 = 20$ because the 3!'s "cancel out". This only works when you are dividing one factorial by another.

WORK



Practice: Working with Factorials

Find the number value of the following:

(a) $3! =$ ______________________________

(b) $4! \times 2! =$ ______________________________

(c) $5! + 3! =$ ______________________________

(d) $3! + 4! + 1! + 2! =$ ______________________________

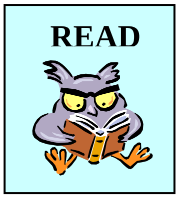
(e) $\frac{8!}{7!} =$ ______________________________

(f) $(12 - 9)! =$ ______________________________

(g) $(8 - 8)! =$ ______________________________

CHECK

With Online Answer Sheet



Permutations and Combinations

An important distinction when figuring sample space size and probability is whether or not the order that the outcomes matters.

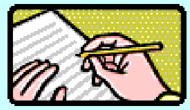
- **Permutation**: the **order DOES MATTER**. For instance when a head followed by a tail is counted as a **HT** while a tail followed by a head is counted as **TH**.

  In **permutation problems**, either the problem will say something like “in order” or “order counts” or the situation by context will tell you that the order counts. For instance, the order of the digits in your school ID number matters. The order of the letters and numbers on your car license plate matters.

- **Combination**: the **order DOES NOT matter.** For instance when a head followed by a tail is counted the same as a tail followed by a head – either one would be **OE** “one of each”.

  In **combination problems**, either the problem will say “in any order” or the situation, by context will tell you that any order is the same “outcome”. For instance, the order in which you are dealt 5 cards does not change the “hand” you are dealt.

WORK



Combination and Permutation Practice

Class Practice: Label each of the situations below as **C** for Combination or **P** for Permutation.

________1. How many ways to get a 3 topping pizza, if there are 12 toppings to choose from and you cannot repeat a topping you have already chosen.

________2. How many ways the digits 1, 2, 3, 4, 5, 6, 7 can be arranged into 7-digit telephone numbers.

________3. How many possible 10-card hands are possible from a deck of 52 cards.

________4. Suppose you know that the combination to the lock you want to open has the following turns: 07, 22, 55. If you do not know what order they go in, how many ways would you have to try to be sure you have tried all the orders using these three turns.

________5. How many ways the answers to a multiple choice question can be arranged if there are 5 different answers: A, B, C, D, and E.

________6. How many 3-color color schemes could be picked at random from a palette of 24 colors, if you are not allowed to repeat a color within the 3 choices.

________ 7. You know your PIN contains the digits 0,9,8,7, and 6 but you can't remember what order. How many different PIN's could there be using these digits?

CHECK

With Online Answer Sheet

Notation and Formulas for Combinations and Permutations

Combination Formula: ${}_nC_r = \dfrac{n!}{(n-r)!\,r!}$

stands for "the number of possible **combinations** (order does NOT count) of **n** things taken **r** at a time".

(1) For example Baskin-Robbins offers 31 different flavors of ice cream. A "true double" is two scoops of ice cream that are two different flavors. If the scoops are placed in a bowl and their position in the bowl does not matter (a scoop of chocolate and a scoop of vanilla is the same as a scoop of vanilla and a scoop of chocolate), then

$n = 31$, the total number of choices of ice cream flavor

$r = 2$, the number of choices you get to make for each bowl

${}_nC_r =$ how many different bowls of two flavors can be made

(2) Find how many ways you can choose 5 books off a shelf with 12 books on it if the order in which you choose the books does **<u>not</u>** matter.

(continues)

Permutation Formula: $_nP_r = \dfrac{n!}{(n-r)!}$

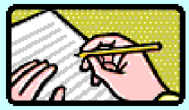
stands for "the number of possible **permutations** (order counts) of n things taken r at a time".

(1) For example, if the "true double" scoops of ice cream from Baskin-Robbins are put on a cone and it matters which flavor is on top, (that is, vanilla on top of a vanilla-chocolate cone is different from chocolate on top of a vanilla-chocolate cone) then the number of different cone choices that can be formed is given by the formula $_nP_r$.

(2) Suppose you need to choose 1st, 2nd, and 3rd place from a group of student artwork. If there are 15 pieces of artwork to judge, how many different ways can 1st, 2nd, and 3rd place be assigned?

Remember the Computation Trick: factorials can be broken down so they cancel out: $\dfrac{16!}{18!} = \dfrac{16!}{18 \cdot 17 \cdot 16!} = \dfrac{1}{18 \cdot 17} = \dfrac{1}{306}$

WORK



Computing Combinations and Permutations

(1) $5! =$

(2) $\frac{3!}{5!} =$

(3) ${}_{10}P_2 =$

(4) ${}_{10}C_2 =$

(5) ${}_{100}C_{98}$

(6) ${}_{100}P_{98}$

(7) (a) Give an example of a situation that would have ${}_{100}C_{98}$ ways to happen.

(b) Give an example of a situation that would have ${}_{100}P_{98}$ ways to happen.

READ

What is Probability?

Experimental Probability: You do the actual probability experiment and record what happens. This is an approximation of the true probability value for the experiment.

Theoretical Probability: You deduce from the characteristics of the sample space and the target set what the probability is. This is the actual value for the probability experiment.

This Chapter deals with Theoretical Probability.

Event: An event is any subset of the sample space. It is a set of individual outcomes.

- When you are asked to list all of the events for a sample space, you are being asked to list all of the possible subsets of the sample space set. Remember that two sets are the same if they have exactly the same elements, regardless of how those elements are ordered.
    - A set with ***n*** elements will have exactly 2^n subsets
    - The empty set is a subset of every set. { }
    - Every set is a subset of itself.

Target Event: The event we want to compute the probability of.

Impossible Event: An event that is not possible.
(**Target Event= empty set**)
The probability of an impossible event is 0.
P(target =empty set) = 0

Certain Event: An event that must happen.
(**Target Event = whole sample space**)
The probability of a certain event is 1.
P(Sample Space) = 1

All probabilities are between 0 and 1 inclusive. (In percents that is 0%-100%)

Probability Assignments
(same as "probability model")

Probability Assignment (sometimes called a probability model): We assign to each individual outcome in the sample space a probability. These assignments must follow two rules:

- the total of all the probabilities of all the events in the sample space must equal 1
- each probability of an individual event must be between 0 and 1 (inclusive).

Is the following a "legal" probability assignment? How can you tell?

Event	A	B	C	D	E
Probability Assignment	0.2	0.1	0.25	0.13	0.32

The following is a legal probability assignment. Find the value of ***x***.

Event	T	U	V	W	X	Y	Z
Probability Assignment	0.1	***x***	***x***	***x***	***x***	0.3	0.1

$S = \{ O_1, O_2, O_3, O_4\}$. $Pr(O_1) + Pr(O_2) = Pr(O_3) + Pr(O_4)$

(a) If $Pr(O_1) = 0.15$, Find $P(O_2)$.

(b) If $Pr(O_1) = 0.15$ and $Pr(O_3)=0.22$, give the probability assignment for this sample space.

Finding "Events" (Target Sets)

The purpose of this type of question is to get you to think about what the target event (target set) will look like for various situations. You have to be able to find the number of elements in the target event (target set} to be able to figure probability.

Consider this situation: 4-question True-False test. Write out the event described by each of the following statements for
S = {all possible strings of 4 letters, where every letter is either T or F}

TTTT	**TTTF**	**TTFF**	**TFFF**	**FFFF**
	TTFT	**TFTF**	**FTFF**	
	TFTT	**FTTF**	**FFTF**	
	FTTT	**TFFT**	**FFFT**	
		FTFT		
		FFTT		

E1: "exactly 2 of the answers given are T"

E2: "at least 2 of the answers given are T"

E3: "at most 2 of the answers given are T"

E4: "The first 2 answers given are T"

(continued)

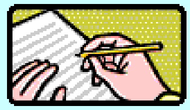
(continued)

Consider the random experiment of drawing 2 cards out of an ordinary deck of 52 cards. The order that the cards are drawn does not matter. Write out the event described by each of the following statements as a set.

E1: “Draw a pair of queens”

E2: “Draw a pair” (A pair is two cards of the same value, either numbers or faces).

WORK



Practice Finding Target Spaces

(1) For the situation “toss a standard die” list the target space for each of the following events:

E1: a number greater than 4

E2: a number less than or equal to 5

E3: an even number

(2) For the situation “toss two standard dice” list the target space for each of the following events:

E1: toss “doubles”

E2: the sum (addition answer) is 7

E3: the sum is 11 or 12

(3) A family plans to have exactly 3 children.

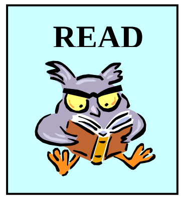
E1: They have all boys

E2: They have exactly 2 girls

E3: They have at least 1 girl.

CHECK

With Online Answer Sheet



Probability Spaces with Equally Likely Outcomes

(text section 15.5)

Equally Likely Outcomes: each individual outcome has an equal probability, for instance

- toss an honest coin (heads and tails are equally likely)
- roll an honest die (1, 2, 3, 4, 5, 6) are all equally likely

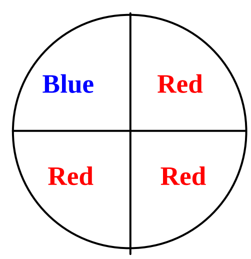
Computing Probabilities <u>when all Outcomes are Equally Likely</u>

- **N =** number of elements in the Sample Space
- **If E is our target event** (remember an event is a subset of the Sample Space), then

$$\textbf{Pr(E)} = \frac{\textit{number of outcomes in set } E}{N}$$

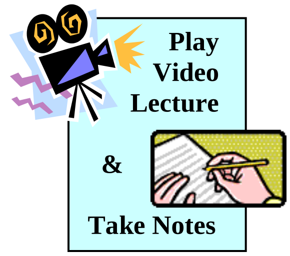
$$= \frac{\text{the number of ways the target event can happen}}{\text{total number of outcomes in the sample space}}$$

For the spinner at the right, each of the four sections is equally likely.



$$\text{Pr(Blue)} = \frac{1 \text{ way to get the target "blue"}}{4 \text{ equally likely possible outcomes}} = \frac{1}{4}$$

$$\text{Pr(Red)} = \frac{3 \text{ ways to get the target "red"}}{4 \text{ equally likely possible outcomes}} = \frac{3}{4}$$



Example 15.22 (page 526 in text) The top card is drawn from a well-shuffled standard deck of 52 cards. What is the probability of drawing an ace?

(a) What is the sample space?

(b) What is the target event?

(c) Are the outcomes in the sample space equally likely? How can you tell?

(d) Pr("top card is an Ace") =

Example 15.25 (page 527 in text) Suppose we roll a pair of honest dice. Remember the sample space for rolling two distinguishable dice looks like this:

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

(a) Pr(11) =

(b) Pr (7) =

(c) Pr (7 or 11) =

(d) Pr(sum <10)

(continues)

Finding “NOT” Probabilities

Notice that since the total probability in a probability assignment = 1, you can find the probability that “Event E doesn’t happen” by taking 1 – Pr(E).

Example: Suppose you roll a die and win if you get anything other than a 6.

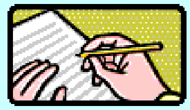
Pr(Not a 6) = 1 –Pr(6) =

Often for probabilities expressed with a negative like this, it is easier to find the other ones and subtract.

Example: Suppose we know that the probability of getting the flu this summer is 0.12. What is the probability of not getting the flu?

Example: If you toss a coin 3 times and record the outcomes in order, what is the probability of “not getting 3 tails”.

WORK



Practice Finding Probabilities

(1) Toss a standard die. Find the following probabilities:

(a) Pr (a number less than 4)

(b) Pr (an even number)

(c) Pr (a number greater than or equal to 2)

(d) Pr (not getting a 4)

(2) Toss two standard dice, one red and one white, recorded (red,white).

(a) Pr (sum of the faces is 4)

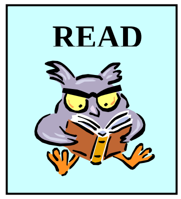
(b) Pr (the red die is 5)

(c) Pr (doubles)

(d) Pr (not getting a 1 on either die)

CHECK

With Online Answer Sheet



Independent Events

(text page 525)

Multi-stage experiment: a probability experiment that repeats a process or follows one probability experiment by another. For example, drawing three cards in a row from a deck, tossing a coin 4 times, or toss a coin and then spin a spinner.

Often the probabilities of events in multi-stage experiments can be computed by computing the probabilities of the separate events. But to do this, we need to pay attention to whether the component events are **Independent** or **Dependent.**

Independent Events: two events are independent if the outcome of one event does not affect the outcome of the other event.

- Roll a red die and a white die. What happens on the red die does not affect what happens on the white die. The red die roll is i**ndependent** of white die roll.

- Draw 1 card from deck. Do not return the card. Then draw another card. On the second draw, you cannot draw the card you drew on the first draw. The card drawn on the second draw is **dependent** on what card was drawn on the first draw. What happens on the first draw affects what can happen on the second draw.

Multiplication Principle for Independent Events: When the events in a multi-stage probability experiment are independent, then we can calculate the probability of the multi-stage event by multiplying the probabilities of separate stages.

Remember: you can only multiply probabilities if the events are Independent.

General rule: "and" usually means multiply the probabilities while "or" means to add the probabilities.

Finding Probabilities Part 2

Example: What is the probability of rolling a die twice, getting a 1 or a 2 on the first roll and getting a 3 on the second roll.

(a) are the rolls of the dice independent?

(b) P(1 or 2) =

(c) P(3) =

(d) P(1or 2 on first roll, followed by a 3 on the second roll) =

(continues)

(continued from previous page)

Example 15.21 (text p. 525)

If we roll an honest die 4 times, what is the probability that we will roll a 1 at least once?

Think of "win" as get at least one roll of 1 out of the 4 rolls.

So a "loss" is not getting any 1's out of the 4 rolls.

All four rolls are independent of each other.

Pr(not getting a 1 on the first roll) =

Pr(not getting a 1 on the second roll) =

Pr(not getting a 1 on the third roll) =

Pr(not getting a 1 on the fourth roll) =

Use the Multiplication Principle to find

Pr(not getting a 1 on first, second, third or fourth roll) =

Use the "NOT" probabilities strategy

Pr (getting a 1 on at least one of the rolls) =

(continues)

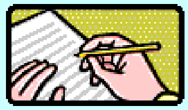
Problem 58, p. 535 Suppose that the probability of giving birth to a boy and the probability of giving birth to a girl are both 0.5. In a family of 4 children, what is the probability that

(a) all 4 children are girls

(b) there are two girls and 2 boys (in any order)

(c) the youngest child is a girl.

WORK



Multi-Stage Probability Practice

Toss a standard die 4 times.

(a) Pr(a 6 on all 4 tosses)

(b) Pr(a value less than 3 on all 4 tosses)

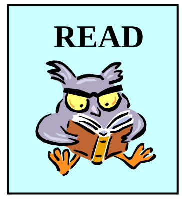
(c) Pr(1, 2, 3, 4 in that order)

(d) Pr(not getting any 1's)

(e) Pr(not getting a 1 on the first toss)

CHECK

With Online Answer Sheet



Odds in Favor of an Event – Odds Against an Event

Odds in favor of an event: The odds in favor of event E are given by the ratio of the number of ways event E can occur to the number of ways which event E cannot occur (all the rest of the Sample Space).

ways to "win": ways to "lose"

Odds against an event: If the odds in favor of event E are ***m*** to ***n***, then the odds against event E are ***n*** to ***m*.**

ways to "lose": ways to "win"

Note: there is a difference between mathematical odds and the "payoff odds" posted by casinos or bookmakers. The connection between payoff odds and actual odds is tenuous at best.

Example: Pr(E) = $\frac{4}{5}$**.** This is a fraction. It means 4 ways to "win" out of 5 ways the event can happen. Notice that the ways to "lose" is equal to $(5-4)=1$

The **odds in favor of event E** are 4:1

The **odds against event E** are 1:4

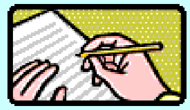
Example: If the **odds in favor** of an event B are 2:3, then the Pr(B) = $\frac{2}{5}$.
Notice that the total number of ways that event B can happen is (2+3) = 5.

Example: If Pr(A)=.35. Since the Pr(not A) is $1-.35=.65$, we can find the

Odds in favor of A = 35:65 (reduces to 7:13)

Odds against A = 65:35 (reduces to 13:7)

WORK



Odds and Probability Practice

(1) The probability of an event T is Pr(T)= 0.7

(a) Find the odds in favor of T

(b) Find the odds against T

(2) The odds in favor of an event W are 2:5.

(a) Find P(W)

(b) Find the odds against W

(3) The odds against an event Q are 3:5.

(a) Find P(Q)

(b) Find the odds in favor of Q.

CHECK

With Online Answer Sheet

End of Chapter Checklist

Before beginning Part 5, be sure you have done the following:

_____1. Worked through all the guided notes
for Part 4: Chances, Probabilities, and Odds

_____2. Worked through all the accompanying online video sessions
for Part 4: Chances, Probabilities, and Odds

_____ 3. Checked all the "WORK" pages with the online answer sheets
for Part 4: Chances, Probabilities, and Odds

_____4. Completed, checked, and corrected the textbook homework
Pages 531-535 #1, 3, 9, 11, 14, 15, 19, 30, 33, 34, 39, 59, 62, 65

_____5. Taken the online practice quiz for Part 4 –Chap 15: Chances, Probabilities, and Odds

Part 5: Normal Distributions

(Chapter 16 in textbook)

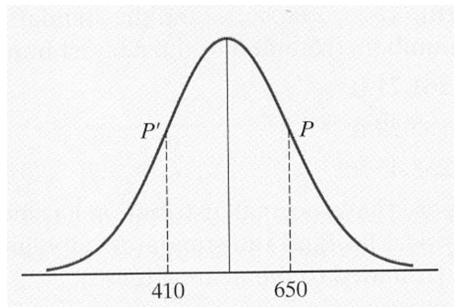
To the student:

- This Part corresponds to Chapters 16 in the textbook
- First, read Chapter 16. Read for the BIG IDEAS and basic meaning. Don't worry about the details. We will do that in the guided notes.
- Then work through these guided notes
- Then do the homework assignment below
  Homework: Pages 558-562 # 1, 3, 12, 15, 16, 25, 26, 31, 41, 47, 48, 49, 53
- Make Arrangements to take Exam 1 over Parts 1-5 by Oct. 8th

Associated Online Resource for Part 5 Guided Notes (GN)		
GN Page	**Type of Online Resource**	**Name of Online Resource**
114	**Video Lecture**	Chapter 16: Normal Distributions
115	Answer Sheet	Practice Interpreting a Normal Curve
117	**Video Lecture**	Standard Scores
118	Answer Sheet	Practice Standard Scores
119-120	**Video Lecture**	The 68-95-99.7% Rule
122-123	Answer Sheet	Practice Interpreting Normal Distributions
124	Chapter 16 **Online Quiz**	Online Quiz
124	**Unit 1 Test over Parts 1-5**	Ratios, Proportions, Percents, and Chap. 13-16

Chapter 16: Normal Distributions

Normal Curves are the graphs of Normal Frequency Distributions
all share these characteristics:



- **Bell-shape:**
  Data bunches up in the middle and drops off at the outside edges
  - Mean = Median = Center
  - Left-Right Mirror Symmetry
  - Two Inflection points: one on each side where the curve "changes bend direction"
  - Distance from point of inflection to vertical axis = σ
  - Q3 $\approx \mu + (0.675)\sigma$
  - Q1 $\approx \mu - (0.675)\sigma$

Mean

Median

Center

Standard Deviation

Inflection points

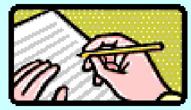
Q1

Q3

- **The 68-95-99.7% Rule**
  - 68% of all the data is between +1 and –1 standard deviations from the mean
  - 95% of all the data is between +2 and –2 standard deviations from the mean
  - 99.7% of all the data is between +3 and –3 standard deviations from the mean

- **Approximately Normal Distributions** have Mean $\approx$ Median

WORK



Practice Interpreting a Normal Curve

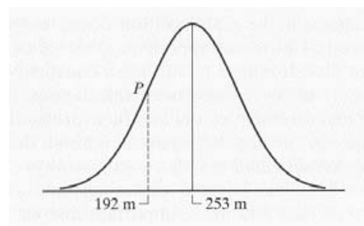
Problem 2 page 558

For the normal distribution described by the curve shown below, find

(a) the mean

(b) the median

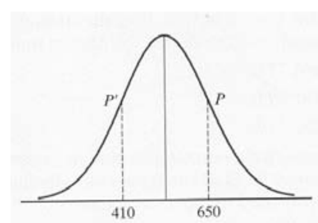
(c) the standard deviation



Problem 4 page 559

In the normal curve in the figure below, P and P′ are the inflection points.
For the normal distribution described by this curve, find

(a) the center

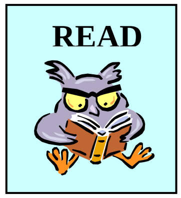


(b) the standard deviation

(c) the first and third quartiles rounded to the nearest point

CHECK

With Online Answer Sheet



Section 16.3 Standardizing Normal Data Sets

(Textbook p. 546)

Bell-shaped curves can differ by vertical and horizontal scale.

We can **Standardize Normal Data Set**s (Data sets that make bell-shaped curves) by relating each data point to how many standard deviations it is from the mean. These standardized scores are called z-scores.

- 1 standard deviation above the mean (M + σ) is standardized as **z** = +1
- 1 standard deviation below the mean (M – σ) is standardized as **z** = –1
- All other data points are converted proportionally using the formula

original data value	$\Rightarrow$	standardized data value
x	$\Rightarrow$	$z = \dfrac{(x-\mu)}{\sigma}$

- Standardized scores can be converted back into the original data values by

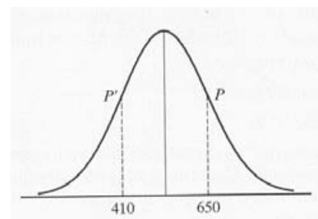
standardized data value	$\Rightarrow$	original data value
z	$\Rightarrow$	$x = \mu + \sigma z$

Standard Scores

My spelling test story.

Formulas: $z = \dfrac{x - \mu}{\sigma}$ and $x = \mu + \sigma z$

On a test with the distribution shown at the right, Maxie scored 650.
What is Maxie's standard score and what does that tell us?



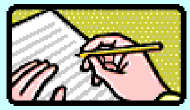
On the same test Trini scored 480. What is Trini's standard score and what does it tell us?

Chris has a z-score of 1.5.

(a) What does this z-score tell us about how Chris did on the test?

(b) Find the test score Chris actually made.

WORK



Standard Score Practice

Example 16.4 (text p. 547)

Consider a normal distribution with mean $\mu = 45\ ft$ and standard deviation $\sigma = 10\ ft$

(a) Find the z-scores for the following data.

Original Data Value	***z*-score**
55 feet	
35 feet	
50 feet	
21.58 feet	

(b) Using the same distribution, find the actual score if the z-score is .85

CHECK

With Online Answer Sheet

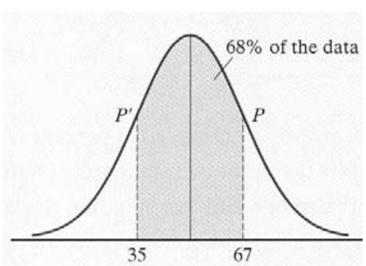
The 68-95-99.7% Rule and Z-Scores

So how does standardizing to z-scores help us?

- It gives us a standard so we can compare distributions
- We know 68% of all data in a Normal Distribution are between $+1\,\sigma$ *and* $-1\,\sigma$
  (Therefore 68% of all data are between the z-scores of -1 and + 1)
- We know 95% of all data in a Normal Distribution are between $+2\sigma$ *and* -2σ
  (Therefore 95% of all data are between the z-scores of -2 and +2
- We know 99.7% of all data in a Normal Distribution are between $+3\sigma$ *and* -3σ
  (Therefore 99.7% of all data are between the z-scores of -3 and +3.

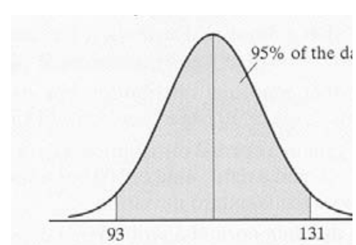
Problem 30 page 560

Find the mean μ and standard deviation σ for the normal distribution described in the following figure:



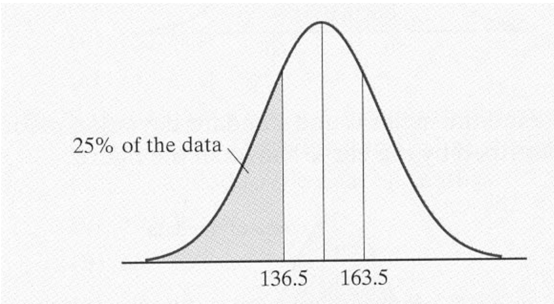
Problem 32 page 560

Find the mean μ and standard deviation σ for the normal distribution described by the curve shown in the figure below.



Problem 34 page 561

Find the mean μ and standard deviation σ for the normal distribution described by the curve shown in the figure below.

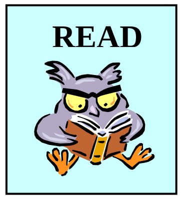


Problem 46 page 561

As part of a research project, the blood pressures of 2000 patients in a hospital are recorded. The systolic blood pressures (given in millimeters) have an approximately normal distribution with mean $\mu = 125$ *and* $\sigma = 13$.

(a) Estimate the number of patients whose blood pressure was 99 millimeters or higher.

(b) Estimate the number of patients whose blood pressure was between 99 and 138 millimeters.



Normal Curves and Computing Margin of Error

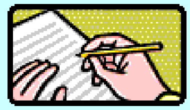
Margin of Error is computed by knowing that the distribution of possible samples of size n will be normally distributed and the standard deviation of that distribution will depend on the size and number of samples taken.

So if you know the size of the samples and how many such samples are taken in a survey or opinion poll, and you know the mean of all of the samples, you can find where 68-95-99.7% of the samples will fall using the 68-95-99.7% rule.

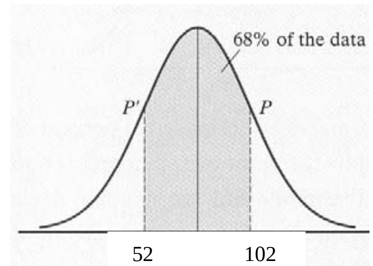
Statisticians can do percents other than these. However, a $\pm 3\%$ Margin of Error usually comes from the 95% rule. That gives you 2.5% of the population above and 2.5% of the population below the "good fit" region. They round these up to 3% above and below to make extra sure they can depend on their confidence interval.

You are not responsible for computing Margin of error, only for understanding how it is related to the idea of a normal distribution.

WORK



Practice Interpreting Normal Distribution Graphs



1. For the normal distribution pictured at the right, determine the following:

    (a) the mean of the distribution = ______

    (b) $\mu + 1\sigma$

    (c) $\mu - 1\sigma$

    (d) $\mu + 2\sigma$ =_________ and sketch its position on the graph

    (e) _________% of all the data points fall between 52 and 102.

    (f) __________% of all the data points fall between 2 and 152.

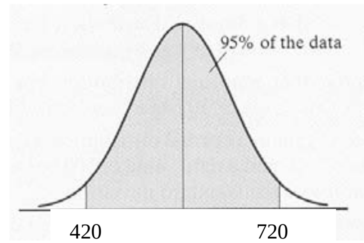
    (g) 25% of all the data points fall below the data value of _____________.

    (h) 50% of all the data points fall below the data value of _____________.

    (i) ____% of all the data points fall below 102.

(continues)

2. For the normal distribution pictured at the right



(a) the median = ________

(b) the standard deviation =__________

(c) the 3rd quartile is ___________

(d) What percent of the data falls below 420?__________

(e) What percent of the data falls above 645? __________

(f) 68% of all data falls between ________ and ________.

(g) $\mu \pm 3\sigma$ = ________________ and _________________

CHECK

With Online Answer Sheet

End of Chapter Checklist

This is the end of the Unit 1.

______1. Worked through all the guided notes
for Part 5: Normal Distributions

______2. Worked through all the accompanying online video sessions
for Part 5: Normal Distributions

______ 3. Checked all the "WORK" pages with the online answer sheets
for Part 5: Normal Distributions

______4. Completed, checked, and corrected the textbook homework
Pages 558-562 # 1, 3, 12, 15, 16, 25, 26, 31, 41, 47, 48, 49, 53

______5. Taken the online quiz over Part 5 – Chap. 16 Normal Distributions

______6. Reviewed Parts 1, 2, 3, 4, and 5 studying for the first major unit exam.

______7. **Exam 1 over Parts 1-5**

______8. **Then** you may begin Unit 2. Unit 2 begins with
Part 6: Spiral Growth in Nature (Fibonnaci Patterns).

Part 6: Spiral Growth in Nature Fibonacci Patterns
(Chapter 9 in textbook)

To the student:

- **This Part corresponds to Chapters 9 in the textbook**
- **First, read Chapter 9. Read for the BIG IDEAS and basic meaning. Don't worry about the details. We will do that in the guided notes.**
- **Then work through these guided notes**
- **Then do the homework assignment below**
  Homework: Pages 329-333 #1-4, 6, 7, 8, 13, 19, 21, 30, 35 - 37, 41, 48

Associated Online Resources for Part 6 Guided Notes (GN)

GN Page	Type of Online Resource	Name of Online Resource
126-127	**Video Lecture**	Finding and Describing Patterns in Sequences
130-131	Answer Sheet	Practice with Sequence Notation
132	**Video Lecture**	The Fibonacci Sequence
133	Answer Sheet	Fibonacci Practice
135-136	**Video Lecture**	Computing Binet's Formula with a Calculator
138-140	**Video Lecture**	Problem 12 page 379 text
141	Answer Sheet	Quadratic Formula Problems
142	**Video Lecture**	Properties of Quadratic Equations with Consecutive
145	**Video Lecture**	Golden Rectangles
146	Answer Sheet	Practice with Golden Rectangles
147-149	**Video Lecture**	Exponential Powers of ϕ
149	Answer Sheet	Exponential Practice
150 top	**Video Lecture**	Using Fibonacci to Approximate ϕ
150 bottom	Answer Sheet	Practice Using Fib to Approximate
151	Answer Sheet	Comparing Approximations
152-153	**Video Lecture**	Scientific Notation
153 bottom	Answer Sheet	Practice Scientific Notation
154	**Video Lecture**	Fibonacci Numbers in Scientific Notation
155	**Video Lecture**	Similar Figures
156	Answer Sheet	Practice with Similar Figures
157-158	**Video Lecture**	Gnomons
159	Answer Sheet	Practice Solving Gnomons
	Online Quiz Chapter 9	**Online Chapter 9 Quiz**

Intro Chapter 9:
Finding and Describing Patterns in Sequences

A number sequence is an ordered list of number values.

Try to find the pattern and fill in the blanks in each of these examples:

(a) 5, 10, 15, 20, ______, ______, _______, . . .

recursive rule: start with ______ and ______________________________

in symbols: ____

(b) 18, 8, -2, -12, ______, ______, _______, . . .

recursive rule: start with ________ and _________________________________

in symbols: ____

(c) 2, 4, 8, 16, ______, _______, _______, . . .

recursive rule: start with _________and ________________________________

in symbols: ____

(d) 80, 40, 20, 10, _______, _______, _______, . . .

recursive rule: start with ________ and ______________________________

in symbols: ____

(continues)

(Video Lecture Continued)

Two Common Types of Sequences

Arithmetic Sequences have a common difference between terms

	Implicit Rule:	**Explicit Formula**
	Start with a_1 And add d each time	$a_n = a_1 + (n-1)d$

5, 10, 15, 20, _____, _____, ______, . . .

18, 8, -2, -12, _____, _____, ______, . . .

Geometric Sequences have a common ratio between terms:

	Implicit Rule:	**Explicit Formula**
	Start with a_1 and multiply by ***r*** each time	$a_n = a_1 r^{(n-1)}$

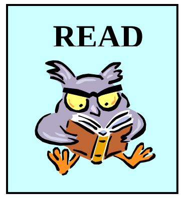
2, 4, 8, 16, _____, ______, ______, . . .

80, 40, 20, 10, _____, _____, ______, . . .

Sequence terms can also be used in algebraic expressions.
For the sequence $S_n = 7n - 2$ find:

(a) S_{3+1} (b) $S_4 + 3$ (c) $5S_5$ (d) $2(S_3 + 1)$

__________ ___________ ____________ ____________



Recursive and Explicit Rules for Sequences

Sequence: An ordered list of numbers that is generated by a rule. Generally sequences are notated with a capital ***S***.

Terms: The numbers in the list are called the terms of the sequence.

Term Number: (usually denoted "N") is the number that tells the term's relative position in the sequence list.

Recursive Rule: the rule that tells how to get the next term of the sequence using earlier terms in the sequence.

Explicit Rule: the formula that generates the sequence from the **<u>term</u>** number.

Recursive rules are usually easier to understand and more convenient when you are writing out the terms of the sequence <u>in order</u>. BUT recursive rules don't help if you want to know the 1000^{th} term without finding the other 999 other terms first.

Example: The odd counting numbers form a sequence: $S_N = \{1, 3, 5, 7, 9, \ldots\}$

The **<u>recursive rule</u>** for the odd counting numbers is:
"Start with 1 and continue by adding 2 each time."

1, 3, 5, 7, 9, 11, . . .

+2 +2 +2 +2 +2

The **<u>explicit rule</u>** for the odd counting numbers is $S_N = 1 + 2(N-1)$

Algebraically simplified this is the same as $= 1 + 2N - 2$

$= 2N - 1$

Continues on next page.

The **recursive rule** produces the **next term** S_{N+1} using the previous term S_N.

In symbols: $S_{N+1} = S_N + 2$

subscripts

The 2 is not a subscript

But the **explicit rule** produces the **current term** S_N using the term number N.

In symbols: $S_N = 2N - 1$

subscript

Not subscripts

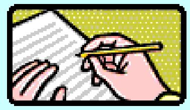
Term Position #	1^{st}	2^{nd}	3^{rd}	**4^{th}**	. . . Nth
Sequence Value	1	3	5	**7**	2N-1
Symbol	S_1	S_2	S_3	**S_4**	S_N

Notice that this is the **explicit** rule

Explicit Rule: 2N-1

Reading the notation: $S_4 = 4th\ term = \underline{2(4) - 1} = 8 - 1 = 7$

WORK



Practice with Sequence Notation

1. (a) Write the first five terms of the sequence that has the recursive rule "start with 2. For each successive term, multiply the previous term by 3".

________, _________, _________, ___________, ________________

(b) Is the sequence above **arithmetic** or **geometric**? __________________

2. (a) Write out the first 5 terms of the sequence $S_n = 5n - 1$ in list form. (Hint, remember you start with n = 1)

______, _______, _________, __________, ________________

(b) What is the numerical value of the 500^{th} term in this sequence?_______

(c) Is the sequence above **arithmetic** or **geometric**? __________________

3. For the sequence ***S*** = 10, 30, 90, 270, . . .

Find the explicit formula and simplify it:

Write the explicit formula in correct sequence notation:

4. Use the sequence rule $S_N = 2^{N-1}$

(a) Find S_3

(b) Find $4S_5$

(c) Find S_{3+4}

5. Use the sequence rule $S_N = 3N - 5$

(a) Find S_{20}

(b) Find $S_4 + 7$

(c) Find $8S_5 + 1$

6. Find the explicit rule in simplified from for each of these sequences:

(a) 90, 92, 94, 96, 98, 100, . . .

(b) 7, 21, 63, 189, 567, . . .

CHECK

With Online Answer Sheet

The Fibonacci Sequence

Fibonacci Sequence: an especially interesting sequence with applications in art, science, nature, and architecture. Your textbook shows pictures of many of these applications of the Fibonacci sequence. Be sure you take time to look at them and read about them.

Your text book uses the letter F for the Fibonacci sequence, so it is notated:

$F = F_1, F_2, F_3, F_4, F_5, ...F_n$

Recursive Rule: $F_1 = 1, F_2 = 1, F_N = F_{N-2} + F_{N-1}$. That is, the first two terms are 1, after that, each successive term is the sum of the two previous terms.

Notice that you can give a formula for a recursive rule. It is just that the formula requires knowing some of the previous values of the sequence.

Write out the first 8 terms of the Fibonacci sequence using the recursive rule:

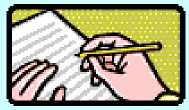
______, ______, ______, ______, ______, ______, ______, ______

Example: $F_{18} = 2584,\ F_{19} = 4181$.

(a) Find F_{20}

(b) Find F_{17}

WORK



Fibonacci Practice

1. Use the recursive rule to find the first 20 Fibonacci Numbers. Remember the recursive rule is that the first two Fibonacci Numbers are both "1" and the next number is always the sum of the previous two numbers.

Term #	Fibonacci #
F_1	1
F_2	1
F_3	
F_4	
F_5	
F_6	
F_7	
F_8	
F_9	
F_{10}	

Term #	Fibonacci #
F_{11}	
F_{12}	
F_{13}	
F_{14}	
F_{15}	
F_{16}	
F_{17}	
F_{18}	
F_{19}	
F_{20}	

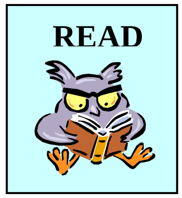
2. $F_{25} = 75{,}025 \text{ } and \text{ } F_{26} = 121{,}393.$

(a) Find F_{24}

(b) Find F_{27}

(c) Find F_{23}

(d) Find F_{28}



Explicit Rule for Fibonacci Sequence (Binet's Formula)

$$F_N = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^N - \left(\frac{1-\sqrt{5}}{2}\right)^N}{\sqrt{5}}$$

You are not responsible for memorizing this formula. You are responsible for being able to use it on your calculator if I give it to you in a problem.

Note: $\sqrt{5}$ is an irrational number.

Use your calculator to calculate $\sqrt{5}$.

You should get something like 2.23606797

But this is only an approximation.

The in the actual decimal value of $\sqrt{5}$ the decimal part never ends and never repeats. Therefore, it is not exactly equal to any fraction. This is what makes $\sqrt{5}$ "irrational". It cannot be written as a fraction, or "ratio", of whole numbers. "Irrational" means "not a ratio".

So when doing computations with $\sqrt{5}$, you have two choices. You can use the symbol $\sqrt{5}$ algebraically, and that will give you an exact solution. Or you can use a decimal approximation for $\sqrt{5}$, which will give you an approximation, not an exact value of the solution.

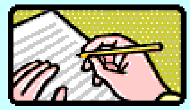
Also, to get a better approximation when using $\sqrt{5}$ in calculations, use $\sqrt{5}$ on your calculator rather than a decimal approximation. It will still give you an approximation, but it will be the best approximation your calculator can do.

Example: on my calculator

$$(2.2)^{20} \approx 7{,}054{,}294 \quad but \quad \left(\sqrt{5}\right)^{20} \approx 9{,}765{,}625$$

In calculations like the above exponentials, using a rounded value rather than the $\sqrt{5}$ makes a HUGE difference (about 2 MILLION), as you can see.

WORK



Computing Binet's Formula with a Calculator

Compute the following with your calculator.

Remember you will still get an approximation. So use the $\approx$ sign in front of your decimal answer.

Use $\sqrt{5}$, not a decimal approximation.

Example of Calculator Keystrokes for F_2 below:

My calculator automatically puts these three left-parentheses in immediately after the square root sign. Look to see if your calculator does this or if you need to enter them yourself.

$$(((1+\sqrt{\ }(5))\div 2)^2-((1-\sqrt{\ }(5))\div 2)^2)\div\sqrt{\ }(5)$$

$$F_2=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^2-\left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}}\approx$$

(you should get something very close to 1, because $F_2 = 1$)

$$F_7=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^7-\left(\frac{1-\sqrt{5}}{2}\right)^7}{\sqrt{5}}$$

(you should get something very close to 13 , because $F_7 = 13$)

(continues)

(continued from previous page)

These are some other calculator problems to practice.

Problem #20 p. 331. Remember this is going to give us approximations because the calculator uses an approximation for $\sqrt{5}$.

Use your calculate to calculate each of the following to 5 decimal places

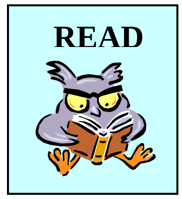
(b) $\left(\dfrac{1+\sqrt{5}}{2}\right)^{10}$

(a) $55\left(\dfrac{1+\sqrt{5}}{2}\right)+34$

(c) $\dfrac{\left(\dfrac{1+\sqrt{5}}{2}\right)^{10}-\left(\dfrac{1-\sqrt{5}}{2}\right)^{10}}{\sqrt{5}}$

CHECK

With Online Answer Sheet



Fibonacci Sequence and Related Number Patterns

Fibonacci Numbers are not only related to patterns in nature, they form fascinating mathematical patterns and properties as well. These will be the subjects of the rest of our study of Fibonacci Numbers.

Pattern #1: Every counting number (Fibonacci or not) can be written as the sum of distinct Fibonacci numbers. "Distinct" here means that you cannot use the same number more than once. Each of the numbers to be added must be a different number from all the rest.

In fact, there is usually more than one way to do it.

(a) Example $F_6 = 8$. Write 8 as the sum of distinct Fibonacci numbers.

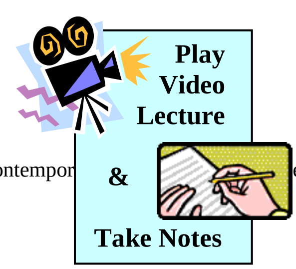
$F_1 + F_3 + F_5$ *OR* $F_4 + F_5$

1 + 2 + 5 **3 + 5**

(b) Write 100 (not a Fibonacci number) as the sum of distinct Fibonacci numbers.

$F_{11} + F_6 + F_4$ *OR* $F_{10} + F_9 + F_6 + F_4$

89 + 8 + 3 **55 + 34 + 8 + 3**



Problem 14, p. 330 text

Consider the following sequence involving Fibonacci numbers:

$$2(2)-3=1$$

$$2(3)-5=1$$

$$2(5)-8=2$$

$$2(8)-13=3$$

a. Look for the pattern and then write down a reasonable choice for the fifth equation in this sequence.

b. Find the subscript that will make the following equation true:
$2(F_?)-F_{15}=F_{12}$

c. Find the subscript that will make the following equation true:
$2(F_{N+2})-F_{N+3}=F_?$

Pattern #2: Any Fibonacci number F_n is equal to the difference of (twice the Fibonacci number two positions after of it) minus (the Fibonacci number three positions after it).

(continues)

Problem 16, p. 330 text "Verify" in mathematics means "show that this is a true statement".

Fact: If we make a list of any 10 consecutive Fibonacci numbers the sum of all these numbers divided by 11 is always equal to the 7^{th} number on the list.

a. Verify this fact is true for the first 10 Fibonacci numbers.

b. Verify this fact for the set of 10 Fibonacci numbers starting with F_5

c. Using F_N at the first Fibonacci number on the list, write this fact as an equation. (This is a notation problem).

(continues)

(continued)

Quadratic Equations that have consecutive Fibonacci numbers as coefficients also have a special property.

First we will look at quadratic equations in general, then at some examples.

Then we will state the Fibonacci property related to Quadratic Equations.

Quadratic Equations: equations of the form $ax^2 + bx + c = 0$.

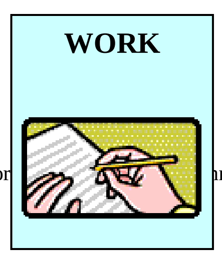
Quadratic equations always have two solutions (sometimes they are identical, sometimes they are complex, but there are always two).

Quadratic Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

You are responsible for memorizing this formula and being able to apply it to solve quadratic equations.

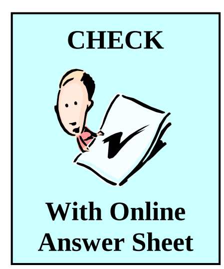
Class Practice: Solve $x^2 = x + 1$ the quadratic formula and then use a calculator to estimate the solutions to 5 decimal places.

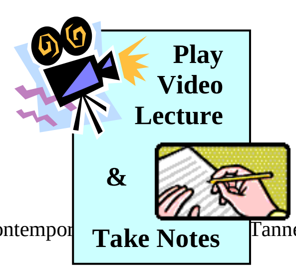
Reminder: first you must algebraically rewrite $x^2 = x + 1$ in the form $ax^2 + bx + c = 0$



1. Show how to use the quadratic formula to solve this equation:
$2x^2 + 5x - 12$

2. Problem 29 page 331 Solve $x^2 = 3x + 1$ using the quadratic formula and then use a calculator to estimate the solution to 5 decimal places.





Properties of Quadratic Equations with Consecutive Fibonacci Coefficients

Pattern #3: In a quadratic equation of the form $ax^2 = bx + c$ with coefficients a, b, and c that are consecutive Fibonacci numbers, the solutions are $-1\ and\ \frac{-b}{a}$.

Problem #32 on page 331

Consider the quadratic equation $21x^2 = 34x + 55$

(a) Without using the quadratic formula, find one of the solutions to this equation. (Hint: One of the solutions is a negative number close to zero)

(b) Without using the quadratic formula, find the other solution to the equation.

READ

The Golden Ratio

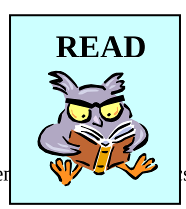
Golden Ratio: the value $\frac{1+\sqrt{5}}{2}$. Your textbook has several good picture-examples of the golden ratio as it appears in art, architecture, and nature.

Notice that $\frac{1+\sqrt{5}}{2}$ is one of the values from Binet's Formula for the Fibonacci sequence (guided notes p. 137). We might expect that there are relationships between the Golden Ratio and Fibonacci numbers.

In mathematics the value of the Golden Ratio is represented by the symbol ϕ (pronounced "fee").

Note the exact value of ϕ is $\frac{1+\sqrt{5}}{2}$. This is an irrational number. As we discussed before, irrational numbers cannot be written exactly as a decimal that ends or repeats or exactly as a fraction.

ϕ is the first letter of the name Pheidras, an ancient Greek sculptor who reputedly used the Golden Ratio in his work. The symbol ϕ is used to stand for the <u>exact value</u> of the Golden Ratio since that value cannot be written exactly as a decimal that ends or repeats.



If we are willing to approximate, however, a good decimal approximation for the Golden Ratio ϕ is 1.618.

Remember the symbol for "approximately equal to" is $\approx$.

"ϕ is exactly equal to $\frac{1+\sqrt{5}}{2}$" is written in symbols: $\phi = \frac{1+\sqrt{5}}{2}$

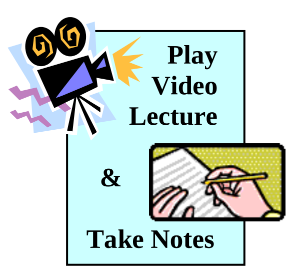
"ϕ is approximately equal to 1.618" is written in symbols: $\phi \approx 1.618$

The distinction between being "exactly equal to" $\phi = \frac{1+\sqrt{5}}{2}$

and being "approximately equal to" $\phi \approx 1.618$

is an extremely important distinction in mathematics.

Remember to use $\approx$ when using an approximation or when you have rounded off.

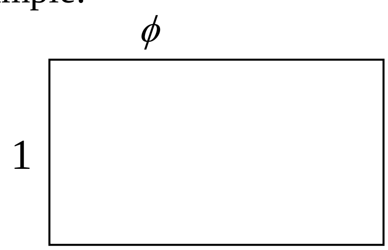


Golden Rectangles

Golden Ratio: Remember $\phi = \frac{1+\sqrt{5}}{2}$ and $\phi \approx 1.618$

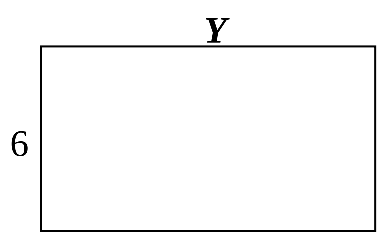
A "Golden Rectangle" is a rectangle that has it's long and short sides related by the Golden Ratio. Specifically, the length of the long side divided by the length of the short side $= \phi$.

Example:

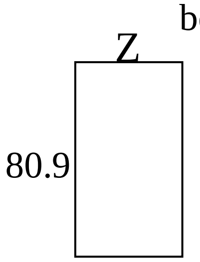


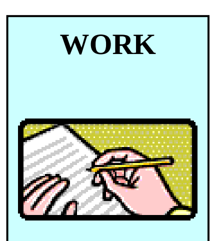
Ratio of $\frac{\text{long side}}{\text{short side}} = \frac{\phi}{1} = \phi$

Example: Find ***Y*** if the rectangle below is a Golden Rectangle.

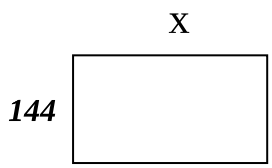


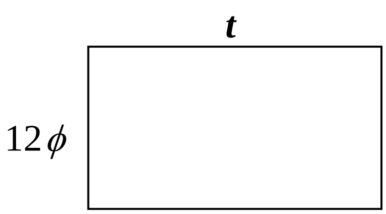
Example: Find ***Z*** to approximated to three decimal places if the rectangle below is a Golden Rectangle.

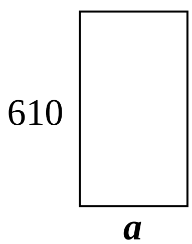


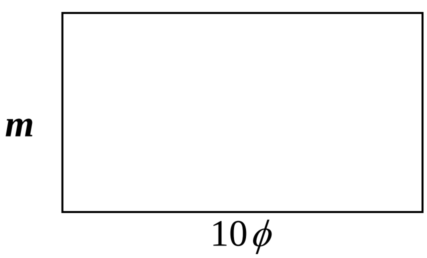


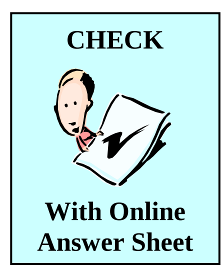
The rectangles below are Golden Rectangles. If the given side is given as a whole number, approximate the length of the other side to the three decimal places. If the given side is given as a multiple of ϕ, then give the exact length (using ϕ) of the other side.

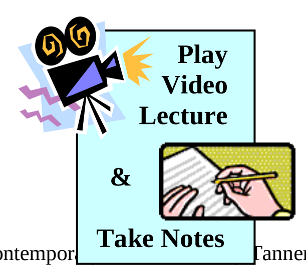












Exponential Powers of ϕ

Let's look again at the equation $x^2 = x + 1$.

This means that ϕ is a solution to the equation $x^2 = x + 1$.

And this means that ϕ^2 is the same thing as taking $\phi + 1$.

We can use this fact to find a very helpful pattern for computing ϕ^N.

(continues)

We will compute ϕ^3, ϕ^4, ϕ^5, ϕ^6, ϕ^7 using what we know about ϕ^2 and some basic algebra.

Then we will see a pattern from which we can make a formula for ϕ^N

$\phi^3 = \phi(\phi^2)$

$\phi^4 = \phi(\phi^3)$

$\phi^5 = \phi(\phi^4)$

$\phi^6 = \phi(\phi^5)$

$\phi^7 = \phi(\phi^6)$

ϕ^N	Value
ϕ^1	
ϕ^2	
ϕ^3	
ϕ^4	
ϕ^5	
ϕ^6	
ϕ^7	

Fibonacci Pattern #4:
Formula for ϕ^n: $\phi^n = F_N\phi + F_{N-1}$

(continues)

(video lecture continues)

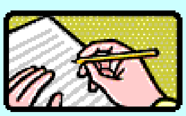
Using the Formula $\phi^n = F_N\phi + F_{N-1}$

Example: Find ϕ^{25} using the formula to find

(a) the exact value

(b) the decimal approximation to five decimal places.

WORK



Now use the formula to find ϕ^{17}

(a) the exact value

(b) the decimal approximation to 5 decimal places.

CHECK

With Online Answer Sheet

Play Video Lecture

&

Take Notes

Using Fibonacci Numbers to Approximate ϕ

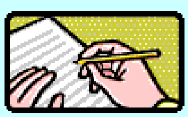
Fibonacci Pattern #5: $\dfrac{F_N}{F_{N-1}} \approx \phi$ and the larger the value of ***N*** the better the approximation.

Problem: Use F_{13} *and* F_{14} to approximate ϕ.

$$F_{13} = 233$$
$$F_{14} = 377$$

Problem 1: Use F_4 *and* F_5 in the formula to approximate ϕ.

WORK

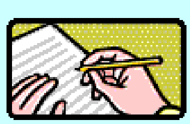


Problem 2: Use F_{19} *and* F_{20} (remember you can compute these easily starting with F_{13} *and* F_{14} given in the example

CHECK

With Online Answer Sheet

WORK



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Comparing Fibonacci Approximations for ϕ

Use your calculator and the formula $\frac{F_N}{F_{N-1}} \approx \phi$ in order to complete the table below.

Term Number N	Fibonacci Sequence Value $F_N =$	Fibonacci Sequence Value F_{N-1}	$\frac{F_N}{F_{N-1}}$
5	5	3	
15	610	377	
20	4181	2584	
21	6765	(hint: this value is in the previous row)	
22			
23			
24			
25			
26			

Note how the approximation for ϕ improves as larger Fibonacci values are used.

CHECK

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Take Notes

Scientific Notation

Scientific Notation is a short-cut way of writing very large or very small decimal numbers.

Scientific calculators automatically switch to scientific notation whenever the value exceeds its maximum number of spaces for digits on the display.

Enter the following into your calculator:

1000^20 and press enter.

The display shows 1 E60. This means:

Change each of the following to hand-written Scientific Notation.

2,000,000 =

145,000,000,000,000,000,000,000,000,000,000,000,000 =

0.000000000000678

23.00008

0.00000000000000000000000000000506

(continues)

Change these values from Scientific Notation to standard decimal notation.

1.42×10^4

7.008×10^{-7}

3.0×10^{-2}

7×10^{8}

WORK

Change to scientific notation:

(a) 6,000,000,000

(b) 0.000076

(c) 707,800,000,000,000,000,000

(d) 0.0000000000000000000000000004567

Change to regular decimal notation.

(e) 2.5×10^{5}

(f) 6.007×10^{-7}

(g) 916.102×10^{4}

(h) $.0000072 \times 10^{8}$

CHECK

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Take Notes

Fibonacci Numbers in Scientific Notation

$F_{99} \approx 1.289 \times 10^{20}$.

Calculate approximations in scientific notation for the following.

Remember $\dfrac{F_N}{F_{N-1}} \approx \phi$ and $\phi = \dfrac{1+\sqrt{5}}{2}$

F_{100}

F_{98}

F_{97}

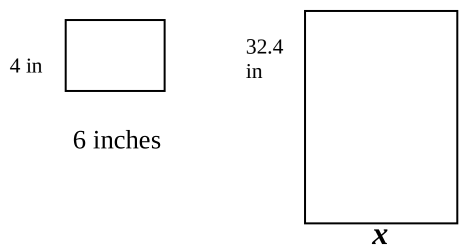
Section 9.3 Similarity

Similar: Two shapes in geometry are similar if they have the same shape.

They may be different sizes with the smaller one being a scaled down version of the larger one.

Fact from Geometry: The lengths of corresponding sides of similar figures are proportional.

Example: These two rectangles are similar (but not drawn to scale, here).



(a) Write out the proportion. Be sure your stack it correctly.

(b) Solve for ***x***.

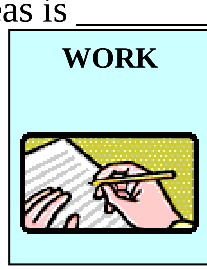
(c) The scale factor is ______.

(d) The perimeter of the smaller rectangle is ______ and the large is ______.

(f) The scale factor for the perimeters is _____________.

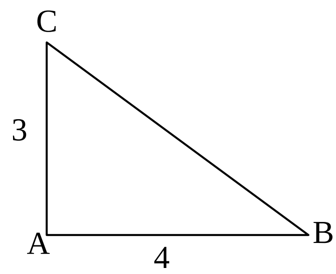
(g) Area of the smaller rectangle is _______ and the area of the larger is _____.

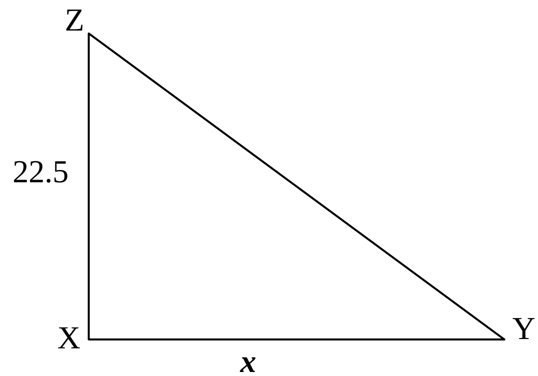
(h) The scale factor for the areas is _________.



Practice Working with Similar Figures

Triangle ABC is similar to triangle XYZ.





1. Write out the correct proportion relating the sides of these two triangles and solve for ***x***.

2. Find the scale factor that relates these two triangles.

3. If the perimeter of triangle ABC is 12 units, what is the perimeter of triangle XYZ?

4. If the area of triangle ABC is 6 square units, what is the area of triangle XYZ?

CHECK

With Online Answer Sheet

Play Video Lecture

&

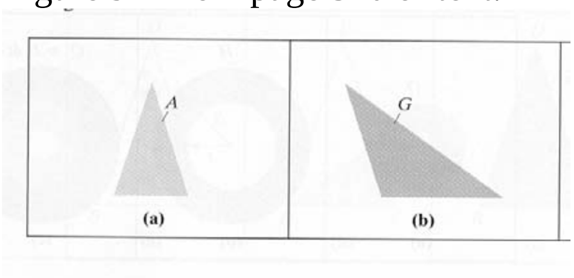
Take Notes

Gnomons

Gnomon: In geometry, "G is a gnomon to the original figure A if G and A connected appropriately without overlap or gap is similar to the original figure A."

Gnomonic Growth: As the organism grows, it remains similar to itself. That is, as it grows, the shape stays the same.

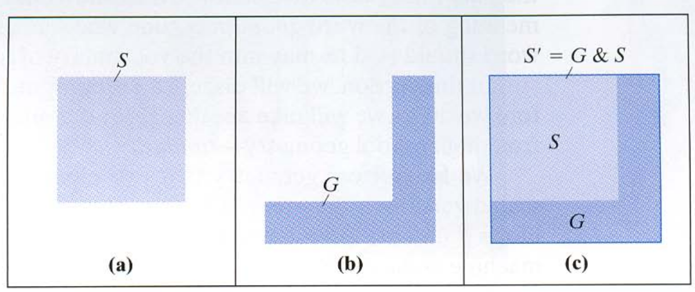
Figure 9-4 from page 320 of text.



Sketch Figure G & A joined together:

Example 9-1, Figure 9-5 p. 320 of text

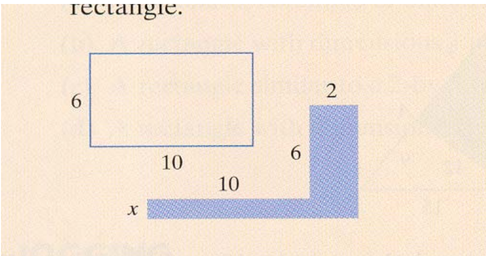
FIGURE 9-2
(a) A square S.
(b) Its gnomon G.
(c) The combined figures form a larger square.



(continues)

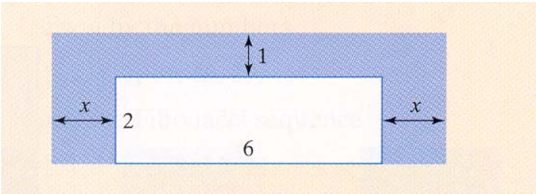
Problem 42, page 332 of text.

Find the value of ***x*** so that the shaded figure is a gnomon to the white rectangle.

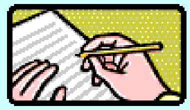


Problem 44, page 332 of text.

Find the value of ***x*** so that the shaded figure is a gnomon to the white rectangle.



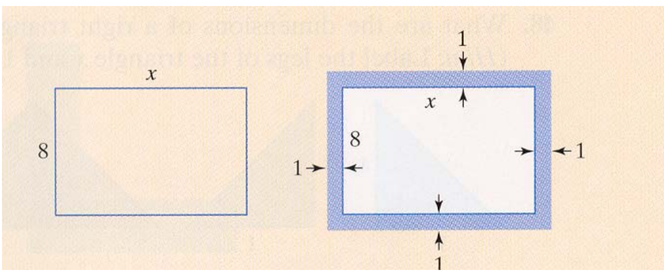
WORK



Practice Solving Gnomons

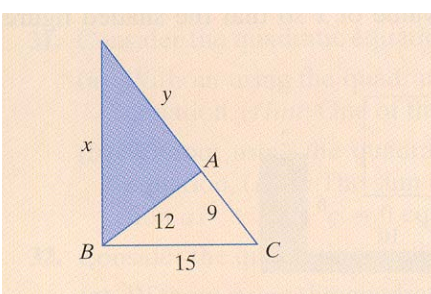
Problem 46, page 332 in text.

Find the value of x so that the shaded “rectangular ring” is a gnomon to the white triangle.



Problem 50, page 333 in text.

Find the value of ***x*** and ***y*** so that the shaded triangle is a gnomon to the white triangle ABC.



CHECK

With Online Answer Sheet

Part 7: The Mathematics of Population Growth

(Chapter 10 in textbook)

To the student:

- **This Part corresponds to Chapters 10 in the textbook**
- **First, read Chapter 10. Read for the BIG IDEAS and basic meaning. Don't worry about the details. We will do that in the guided notes.**
- **Then work through these guided notes**
- **Then do the homework assignment below**

  Homework: Pages 364-369 #1, 5, 9, 10, 13 - 15, 19, 22, 25, 29, 30, 35, 36, 45, 47, 55, & 56

Associated Online Resources for Part 7 Guided Notes (GN)		
GN Page	**Type of Online Resource**	**Name of Online Resource**
162	Answer Sheet	Bacteria Example
163	**Video Lecture**	Fibonacci's Rabbits
164	Answer Sheet	Class Practice P. 167
165-166	**Video Lecture**	Problem 6 on page 420 of text
167-168	Answer Sheet	Practice Finding Sums Linear Sequences
169	Answer Sheet	Exponential Example
170-172	**Video Lecture**	Some Other Exponential Examples
173	Answer Sheet	Money Growth
174	**Video Lecture**	Example 10.13 from Text
176-177	**Video Lecture**	Growth Rate for Period
178	Answer Sheet	Problem 42 on Page 424
	Online Quiz	**Online quiz over Chapter 10**
Exam 2 - Chapters 9-10 and key topics in Chapters 13-16		

READ

Section 10.1: The Dynamics of Population Growth

In this chapter we will study about the mathematics of calculating population growth. There are many different formulas for calculating population growth because different types of organisms have different characteristics that affect how the total population increases, decreases, or stabilizes. Since, in this class, we are interested in the mathematics, rather than the biology, we will begin by looking at some simple types of population growth and their formulas and then advance to the mathematically more complex formulas.

Population sequence: a list that starts with an initial population, and lists the population at the end of each successive unit of time.

Your textbook uses the symbol P_0 for the initial population. The zero indicates that this is the "zero" generation, the generation before any offspring are born. Then your textbook uses the symbols $P_1, P_2, P_3, \ldots, P_N$ for the rest of the population sequence. Notice that this makes P_N is the size of the population after N generations of offspring are born.

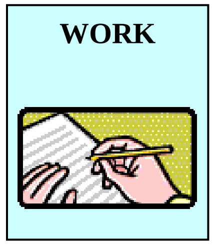
> Situation Example: the beginning population is 100 and every day 2 more are born with no deaths.

Resulting Population Sequence:

P_0	P_1	P_2	P_3	P_4	. . .	P_N
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		$\downarrow$
100,	102,	104,	106,	108	. . .	?

The formulas in this chapter will give use the size of the population in the Nth generation.

Warning: Some of these formulas will be very similar to the formulas we learned in the previous chapter for arithmetic and geometric sequences. But the symbols used will be the symbols that represent **generations** rather than simply terms. We always start with the "zero generation" in these formulas rather than the "first term" like we did in the previous chapter, because we are computing populations after N generations of offspring.



Bacteria Example: suppose a bacteria population doubles every hour. If the initial culture has 10 of the bacteria organisms at time = 0, fill out the following table:

Elapsed Time (in hours)	Population Sequence Symbol	Population at the beginning of that generation
0		
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		
14		
15		
16		

Fibonacci's Rabbits

Fibonacci's Rabbits: A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

Let P_0= the initial population of 2 rabbits, a male and a female.

	Initial population	First Generation	Second Generation	Third Generation	Fourth Generation	Fifth Generation	Sixth Generation
Elapsed Time	0	1 month	2 months	3 months	4 months	5 months	6 months
Number of baby pairs	1	0	1				
Number of mature pairs	0	1	1				
Total number of pairs	$P_0 = 1$	$P_1 = 1$	$P_2 = 2$				

After 1 year (= 12 months = 12 generations)

we will have P_{12} pairs = ________ pairs =_____________total rabbits.

This is a classic problem in mathematics. Notice that this particular population problem results in the Fibonacci Sequence. There are other real-world population problems that also result in the Fibonacci Sequence.

READ

Section 10.2 The Linear Growth Model

Linear Growth Model is any sequence that has a common difference between terms. (This means we are adding or subtracting the same amount, each time, from one term to the next consecutive term).

Example: The population sequence 5, 10, 15, 20, 25, . . .
is an example population following a linear growth model.

Formula for a Linear Population Growth Model: $P_N = P_0 + N \times d$ where

P_0 is the initial population

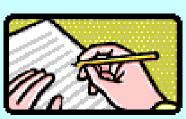
d is the constant difference

N is the number of generations

P_N is the population after N generations of offspring

Notice that this is very similar to the arithmetic sequence formula we learned in the previous chapter except that the symbols now reflect **population and generation terminology.**

WORK



Class Practice: Consider population sequence 200, 194, 188, 182, 176, . . . This represents a declining population in which the number of deaths each generation is greater than the number of births.

(a) What is the common difference ***d***?

(b) What is P_0?

(c) What is the explicit formula for this sequence?

(d) What is P_{12}?

Play Video Lecture

&

Take Notes

CHECK

With Online Answer Sheet

Problem #6 on Page 365 of text

Consider a population that grows according to a linear growth model. The population in the fifth generation is $P_5 = 37$ and the population in the seventh generation is $P_7 = 47$.

(a) Find the common difference $\boldsymbol{d}$

(b) Find the initial population P_0

(c) Find an explicit formula of the population sequence.

(d-extra) Find P_{100}

(continues)

Adding Terms of an Arithmetic Sequence

Formula: for an arithmetic sequence = $A_0,\ A_1,\ A_2,\ \ldots\ A_{N-1}$

The sum of the first N terms (notice that if the initial value is A_0, then the Nth term is A_{N-1}) =

$$Sum = \frac{(\mathit{first\ \ term}\ +\ \mathit{last\ \ term})\times(\mathit{number\ \ of\ \ terms})}{2} = \frac{(A_0 + A_{N-1})\times N}{2}$$

Example 10.7 Compute the sum: 4 + 13 + 22 + 31 + 40 + . . . + 922

(a) Find N. Remember 922 = P_{N-1}

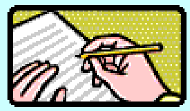
(b) Find the sum.

Practice: for the Sequence $S_N = 4 + 3(N-1)$

(a) Write out the terms from S_{20} up to S_{30} inclusive.

(b) Find the sum of the terms from S_{20} up to S_{30} inclusive.

WORK



Practice Finding Sums of Linear (Arithmetic) Sequences

1. Show how to compute the following sum using the formula we just studied.

$$350 + 353 + 356 + 359 + 362 + 365 + 368 + 371 + 374 + 377 + 380 =$$

2. Show how to compute the following sum using the formula we just studied:

$$63 + 75 + 87 + 99 + 111 + \ldots + 399 =$$

3. For the Sequence $S_N = 8 + 9(N-1)$

   (a) Write out the terms from S_{15} up to S_{25} inclusive.

   (b) Find the sum of the terms from S_{15} up to S_{25} inclusive.

(continues on next page)

4. A population follows a linear growth model. The 4^{th} generation population is 200 and the 9^{th} generation population is 325.

   (a) Find the common difference ***d*.**

   (b) Find the initial population (population in generation zero)

   (c) Write out the formula for this population sequence.

   (d) Find the population in the 500^{th} generation.

5. A population grows according to a linear growth model. The initial population is 1000. The population increases by 214 each month.

(a) Write the formula for the population sequence.

(b) Find the total population after 2 years, assuming all individuals survive.

CHECK

With Online Answer Sheet

READ

Section 10.3 The Exponential Growth Model (Geometric Sequence)

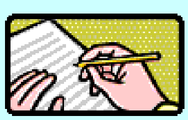
Exponential Growth Model: is any sequence that has a common multiplier between consecutive terms of the sequence. (This means we are multiplying by the same amount each time, from one term to the next consecutive term).

Example: 7, 14, 28, 56, 112, 224, 448, . . .
in this example the common multiplier (or ratio) $r = 2$

Exponential Formula (Explicit): $P_N = P_0 \times r^N$ where

P_0 is the population in generation zero
r is the common multiplier (ratio)
N is the number of generations of offspring
P_N is the population after N generations of offspring

WORK



Exponential Example: A population follows an exponential growth model. The beginning population is 17, and with each generation the population is 5 times as much as it was in the previous generation.

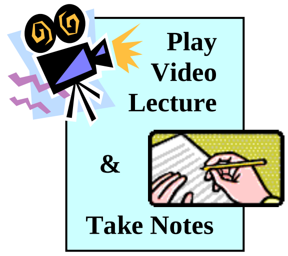
(a) Write out the first 6 terms of this population sequence.

(b) Write the formula for this population sequence.

(c) Compute the population after 12 generations.

CHECK

With Online Answer Sheet



Some Other Exponential Examples

A population grows according to an exponential growth model. The initial population is $P_0 = 8$ and the common ratio is r = 1.5

(a) Find P_1

(b) Find P_9

(c) Give an explicit formula for the population sequence.

(continues)

Adding Terms of a Geometric Sequence

For a geometric sequence with first term a *and common ratio* r**,** the sum of the first N terms is found with the following formula:

$$a + ar + ar^2 + \ldots + ar^{N-1} = \frac{a(r^N - 1)}{r - 1}$$

where a is the first term (or initial population, generation zero)
r is the common ratio
N is the number of terms of the sequence being added counting the first term as term 1.
(yes, your book is inconsistent about what they use for the symbol for the first term of a sequence)

Class Practice: For the geometric sequence
8, 24, 72, 126, 649, 1944, 5832, . . .

(a) Find the recursive formula for the sequence:

(b) Find the explicit formula for the sequence

(c) Find the sum of the first 10 terms of the sequence.

(continues)

Example 10.16 from page 355 of text: An infectious disease first occurs in the year 2000 (year =0), when a total of 5000 cases of the disease were reported in the United States. Epidemiologists estimate that until a vaccine is developed, the virus will spread at a 40% annual rate of growth, and it is expected that it will take at least 10 years to develop a vaccine. Under these assumptions, how many estimated cases of the disease will occur in the United Stated over the 10 year period 2000-2009?

READ

Money Growth

A population does not have to be of people or animals. It can be of something like money, if is in a situation where it is increasing in amount at a predictable rate.

Compounding Rule: Annual Compounding (compounds once a year):

$$P_N = P_0 \times (1+i)^N \text{ where}$$

P_N = the principal,

i= annual interest,

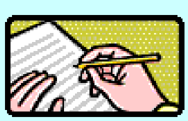
N= number of years.

General Compounding Formula: **(compounds k times a year)**

$$P_N = P_0 \times \left(1+\frac{i}{k}\right)^{N\times k} \text{ where } \begin{aligned} P_N &= \text{Balance in the account after N years} \\ P_0 &= \text{Initial Deposit} \\ i &= \text{Annual interest rate in decimal form} \\ k &= \text{Number of compounding periods in 1 year} \end{aligned}$$

Example

WORK



Suppose you deposit \$367.51 in a savings account that pays an annual interest rate of $9\frac{1}{2}$% a year, and you leave both the principal and the interest in the account for a full 7 years. How much money will there be in the account at the end of 7 years?

CHECK

With Online Answer Sheet

Example 10.14 in text on p. 353

Suppose we find a bank that pays 10% annual interest with the interest compounded monthly. If we deposit $1000 and leave the deposit and the interest in the account, how much will there be in the account at the end of 5 years?

READ

Section 10.4 The Logistic Growth Model

Logistical Growth Model: The rate of growth of the population is directly proportional to the amount of "elbow room" available in a population's habitat.

Lots of "elbow room" means a high growth rate.

Little "elbow room" means a low growth rate, possibly less than 1. Remember that with a growth rate less than one, the population numbers are declining.

Carrying Capacity of the Habitat: a constant that population biologists call C, which is the total saturation point of the habitat. There is no room for growth beyond a population of C.

Growth rate for period

$N = R(C - P_N)$ where C is the carring capacity of the habitat

P_N = current population size and R = contant of proportionality of population

Logistic Equation: $P_{N+1} = r(1 - P_N)P_N$

r is called the growth parameter which depends on both the original growth rate R and the habitat's carrying capacity C.

p_0 = the initial population given as a fraction of the habitat's carrying capacity

restrictions: p_0 should always be between 0 and 1
r should be between 0 and 4

Growth rate for period

$N = R(C - P_N)$ where C is the carring capacity of the habitat

P_N = current population size and R = contant of proportionality of population

Logistic Equation: $P_{N+1} = r(1 - P_N)P_N$

r is called the growth parameter which depends on both the original growth rate r and the habitat's carrying capacity C.

p_0 = the initial population given as a fraction of the habitat's carrying capacity

restrictions: p_0 should always be between 0 and 1
r should be between 0 and 4

Example 10.18 from text on p. 359

We start a fish farming business for a species of trout that has growth parameter r =2.5. If we stock the pond habitat with 20% of its carrying capacity,

(a) how many fish will we have after the first breeding season?

(b) how many fish will we have after the second breeding season?

(c) how many fish will we have after the third breeding season?

(d) How many fish will we have after the fourth breeding season?

(continues)

Problem 46 on page 367 in text

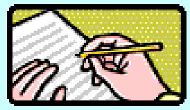
A population grows according to the logistic growth model, with growth parameter r = 0.6. Starting with an initial population given by $p_0 = 0.7$

(a) find p_1

(b) find p_2

(c) Determine what percent of the habitat's carrying capacity is taken up by the third generation.

WORK



Problem 50 on page 368

A population grows according to the logistic growth model, with growth parameter r=1.5.
Starting with an initial population given by $p_0 = 0.8$,

(a) find the values of p_1 *through* p_{10}

(b) What does the logistic growth model predict in the long term for this population?

CHECK

With Online Answer Sheet

End of Chapter Checklist

Before beginning the next Part be sure you have done the following:

_____1. Worked through all the guided notes for
Part 7: The Mathematics of Population Growth

_____2. Worked through all the accompanying online video sessions for
Part 7: The Mathematics of Population Growth

_____ 3. Checked all the "WORK" pages with the online answer sheets for
Part 7: The Mathematics of Population Growth

_____4. Completed, checked and corrected all the homework problems from Chapter 10 in the text **Pages 364-369 #1, 5, 9, 10, 13 - 15, 19, 22, 25, 29, 30, 35, 36, 45, 47, 55, & 56**

_____5. Completed and turned in the **online quiz #7** for
Part 7: The Mathematics of Population Growth

_____6**.** Reviewed and studied for second unit exam over Parts 1-7

_____7. **Exam 2**

Part 8: The Mathematics of Voting

(Chapter 1 in textbook)

To the student:

- **This Part corresponds to Chapter 1 in the textbook**
- **First, read Chapter 1. Read for the BIG IDEAS and basic meaning. Don't worry about the details. We will do that in the guided notes.**
- **Then work through these guided notes**
- **Then do the homework assignment below**
  Homework: Pages 30-38 #1, 4, 6, 13, 19, 20, 24, 27, 34, 35, 36, 41, 44, 49, 51, 59

Associated Online Resources for Part 8 Guided Notes (GN)		
GN Page	**Type of Online Resource**	**Name of Online Resource**
184	**Video Lecture**	Preference Schedule MAS
185	Answer Sheet	Problem 2 Page 28 of Text
186	Answer Sheet	Problem 8 Page 30 of Text
188	Answer Sheet	Revisiting the MAS Results
190	**Video Lecture**	The Bowl Game Example
191	**Video Lecture**	Problem 14 Page 31 Text
192	**Video Lecture**	The Borda Count Method
193	Answer Sheet	Problem 20 Page 33 Text
194	Answer Sheet	Problem 24 Page 34 Text
196	**Video Lecture**	Using Plurality with Elimination
197	Answer Sheet	Problem 34 on Page 35 Text
198	**Video Lecture**	Monotonicity Criterion
200	**Video Lecture**	MAS Election with Pairwise Comparison
201	**Video Lecture**	Example of Violation of Independence
202	Answer Sheet	Problem 38 Page 36 Text
Online Quiz Chapter 1		

READ

The Mathematics of Voting -- Introduction

Voting Theory: the mathematics of the intricacies and subtleties of how voting is done and the votes are counted. In the early 20th century, social scientists and mathematicians working on "social choice" theory examined the methods of democratic voting and their related advantages and disadvantages. In 1949 the conclusion was summarized in Arrow's Impossibility Theorem.

Arrow's Impossibility Theorem: It is mathematically impossible for a democratic voting method to satisfy all of the fairness criteria.

Fairness Criteria (we will study more about these as we go through the chapter):

- **Majority Criterion**: if there is a choice that has a majority of the first-place votes, then that choice should be the winner of the election

- **Condorcet Criterion**: If there is a choice that is preferred by the voters over each of the other choices, then that choice should be the winner of the election.

- **Monotonicity Criterion:** If choice X is a winner of an election and, in a reelection, all the changes in the ballots are changes favorable only to X, then X should still be a winner of the election.

- **Independence-of-Irrelevant-Alternatives Criterion:** If choice X is a winner of an election, and one (or more) of the other choices is disqualified and the ballots recounted, then X should still be a winner of the election.

Voting Methods You are Responsible From this Chapter:

- Plurality Method
- Borda Count
- Instant Runoff Voting (Plurality-with-Elimination)
- Pairwise Comparison (Copeland's Method)

- Extended Ranking Methods

Recursive Ranking Methods

READ

Section 1.1 Preference Ballots and Preference Schedules

Every election has:

- A set of **voters**
- A set of **candidates or choices**
- **Ballots** on which voters indicate their vote.

Preference Ballot: a ballot that asks voters to rank candidates in order of preference

Linear Ballot: a ballot in which ties are not allowed.

Preference Schedule: A table that shows the frequencies of occurrence of each of the actual outcomes of a preference ballot (each way a voter could have ranked the candidates).

Transitivity of Individual Preferences: If a voter prefers A to B and B to C, then it follows automatically that this voter must prefer A to C. Therefore:

- If we need to know which candidate a voter would vote for if it came down to a choice between just two candidates, all we have to do is look at which candidate was placed higher on that voter's ballot.

- If a candidate is eliminated, this does not affect the ***relative preference*** of the voter. The remaining preferences stay in order with the ones rated below the eliminated name moving up 1 ranking.

Majority Rule: In a democratic election between two candidates, the one with more than half of the votes wins.

Plurality Method: The winner is the one with the most first-place votes. Frequently when there are more than 2 candidates, no one candidate receives more than half of the vote, and so the winner, with the most votes, wins by **plurality rather than majority**.

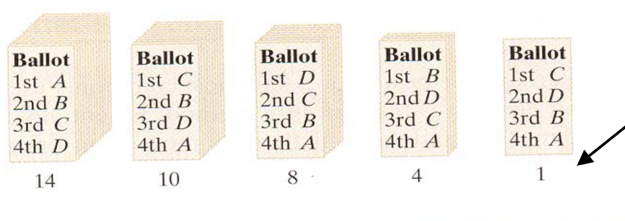
READ

Example 1.1 page 4 of Text

The student organization MAS holds an election for president. There are four candidates: Alisha (A), Boris (B), Carmen (C), and Dave (D). All 37 members of the club vote by means of a ballot indicating his or her first, second, third, and fourth choice.

There are $4! = 4 \times 3 \times 2 \times 1 = 24$ different ways a voter ***could rank*** the four candidates when ties are not allowed.

It turns out in the MAS election there are only 5 ways that actually occurred in this election:



When the ballots are sorted, these are the frequencies of the 5 ballot rankings that occurred.

There should be 37. Are there?

In summary, we can make a **Preference Schedule** for the MAS Election. This is the simplest and most compact way to summarize the voting in an election based on preference ballots.

Creating a Preference Schedule

<u>Step 1:</u> Down the column list 1^{st} , 2^{nd}, 3^{rd}, 4^{th} . . . choices

<u>Step 2:</u> Across the top put the number of ballots that came in that way.

<u>Step 3:</u> Inside the table put the way those ballots came in 1^{st}, 2^{nd}, 3^{rd}, 4^{th} . . . choice

Preference Schedule for MAS Election

# of Voters	14	10	8	4	1
1^{st} Choice	**A**	**C**	**D**	**B**	**C**
2^{nd} Choice	**B**	**B**	**C**	**D**	**D**
3^{rd} Choice	**C**	**D**	**B**	**C**	**B**
4^{th} Choice	**D**	**A**	**A**	**A**	**A**

Preference Schedule for MAS Election					
# of Voters	14	10	8	4	1
1st Choice	**A**	**C**	**D**	**B**	**C**
2nd Choice	**B**	**B**	**C**	**D**	**D**
3rd Choice	**C**	**D**	**B**	**C**	**B**
4th Choice	**D**	**A**	**A**	**A**	**A**

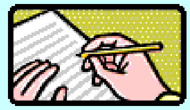
(a) How many voters submitted ballots that ranked the candidates in the order B, D, C, A?

(b) How many voters ranked candidate A in first place?

(c) How many voters ranked candidate C in first place?

(d) How many candidates ranked candidate A in 4^{th} place?

WORK



Problem 2 Page 31

The Latin club holds an election to choose its president. There are three

	The 11 separate ballots appear as columns below.										
1st Choice	C	A	C	A	B	C	A	A	C	B	A
2nd Choice	A	C	B	B	C	B	C	C	B	C	B
3rd Choice	B	B	A	C	A	A	B	B	A	A	C

(a) Make a preference schedule for this election:

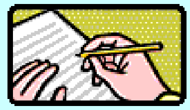
(b) How many first-place votes are needed for a majority?

(c) Which candidate has the most first-place votes? Is it a majority or a plurality?

CHECK

With Online Answer Sheet

WORK



Problem 9 on Page 32 of Text (Modified)

#Voters	**47**	**36**	**24**	**13**	**5**
A	1	2	5	2	4
B	3	1	2	4	1
C	2	4	3	1	5
D	5	3	1	5	2
E	4	5	4	3	3

Rewrite the preference schedule above in the conventional format used in the book.

CHECK

With Online Answer Sheet

READ

Section 1.2 The Plurality Method

Plurality Method: The candidate with the most first-place votes wins. Notice that the only information we use from the ballots are the votes for first place – nothing else matters.

Majority Criterion: If a choice receives a majority of the first-place votes in an election, then that choice should be the winner of the election. This is a criteria of a fair and democratic election. The majority criteria is violated <u>only when there IS</u> a candidate with the majority of the vote, but that candidate does not win.

- If a candidate has plurality, does that mean the candidate also has majority?

  No. When there are more than 2 candidates, it is possible for a candidate to have plurality, but none of the candidates have more than ½ of all the votes.

- If a candidate has majority, does that mean the candidate also has plurality?

  Yes. If the candidate has more than ½ of all the votes, that candidate
  has more of the votes than any other candidate.

- If a candidate wins by plurality, can the election have violated the majority criterion?

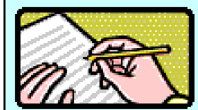
  No. Because the majority criterion is only invoked IF there is a candidate that has received more than ½ of the votes.

What is wrong with the plurality method?

- It fails to take into consideration the voters' preferences other than first choice
- It can therefore lead to some very bad election results.
- It can violate the Condorcet Criterion (next topic)

It can be easily manipulated by insincere voting (next topic)

WORK



Revisiting the MAS Election Results

Revisiting the MAS election results:

Preference Schedule for MAS Election					
# of Voters	14	10	8	4	1
1st Choice	**A**	**C**	**D**	**B**	**C**
2nd Choice	**B**	**B**	**C**	**D**	**D**
3rd Choice	**C**	**D**	**B**	**C**	**B**
4th Choice	**D**	**A**	**A**	**A**	**A**

(a) Who wins by the plurality method? Why?

(b) What percent of the voters ranked candidate A **last**?

(c) What argument could be made that candidate B should be the winner?

CHECK

With Online Answer Sheet

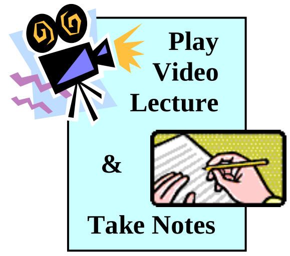
READ

Condorcet Criterion

Condorcet Criterion: If there is a choice that in head-to-head comparisons is preferred by the voters over each of the other choices, then that choice should be the winner of the election.

Condorcet Candidate: A candidate who wins in every head-to-head comparison against each of the other candidates is called the Condorcet Candidate. The Condorcet criterion says that when there is a Condorcet candidate, then that candidate should be the winner. When there is no Condorcet candidate, the Condorcet criterion does not apply.

Insincere Voting: When a voter or block of voters change the true order of their preferences in an effort to influence the outcome of the election by AGAINST a certain candidate. Often used when the voters actually favor a “third party” candidate who has no chance of winning.



The Bowl Game Example

Example 1.3 from text p. 7:
The band must choose which of the five bowl games to march at: Rose Bowl (R), Hula Bowl (H), Cotton Bowl (C), Orange Bowl (O), and Sugar Bowl (S) .

(a) Which Bowl game wins by the plurality method?

(b) "head-to-head" comparison Hula Bowl tow Rose Bowl:

Preference Schedule for Band Choice of Bowl Game Trip

# Voters	49	48	3
1st	R	H	C
2nd	H	S	H
3rd	C	O	S
4th	O	C	O
5th	S	R	R

(c) "head-to-head" comparison of Hula Bowl to Cotton Bowl:

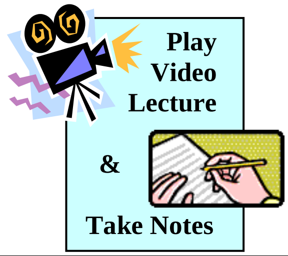
(d) "head-to-head" comparison of Hula Bowl to Orange Bowl

(e) "head-to-head" comparison of Hula Bowl to Sugar Bowl

(f) Is there a Condorcet candidate in this election?

(g) By the Condorcet Criterion, which Bowl Game should the Band march at? What does this represent in terms of the Band's Preferences?

(h) Suppose the three band members who voted C, H, S, O, R above are extremely allergic to lots of kinds of flowers and could not go if the Rose Bowl is selected. IF these three band members had done an informal poll ahead of time and discovered that the Cotton Bowl had no chance of winning, how might they manipulate the election against the Rose Bowl?

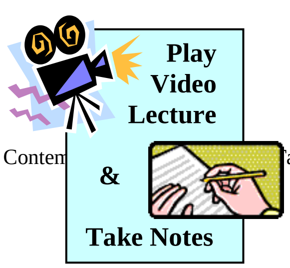


Problem 14 from page 32 Text

An election with 4 candidates (A, B, C, D) and 150 voters is to be decided using the plurality method. After 120 ballots have been recorded, A has 26 votes, B has 18 votes, C has 42 votes, and D has 34 votes.

(a) What is the smallest number of remaining 30 votes that B must receive to <u>guarantee</u> a win for B? Explain?

(b) What is the smallest number of the remaining 30 votes that D must receive to guarantee a win for D? Explain?



Section 1.3 The Borda Count Method

Borda Count Method: Each place on a preference ballot is assigned points. In an election with N candidates, we give 1 point to last place, 2 points for next-to-last place, . . . up to N point for 1st place. The points are tallied for each candidate separately and the candidate with the highest total is the winner.

What's Good about the Borda Count Method:

- It takes into account ALL of the preference rankings and produces a winner that is the best compromise candidate.

What's Wrong with the Borda Count Method?

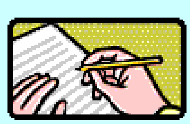
- It can violate the majority criterion for a fair and democratic election: a candidate with the majority of the first place votes can lose the election with this counting method.
- It can also violate the Condorcet criterion for a fair and democratic election: a candidate who wins each "head-to-head" can still lose the election.

MAS Election Using Borda Count

Who wins the MAS Election by Borda Count?

Preference Schedule for MAS Election					
# of Voters	14	10	8	4	1
1st Choice	**A**	**C**	**D**	**B**	**C**
2nd Choice	**B**	**B**	**C**	**D**	**D**
3rd Choice	**C**	**D**	**B**	**C**	**B**
4th Choice	**D**	**A**	**A**	**A**	**A**

WORK



Problem #20 on Page 33 of Text

The members of the Tasmania State University soccer team are having an election to choose the captain of the team from among four seniors: Anderson (A), Bergman (B), Chou(C), and Delgado (D). The reference schedule for the election is given in the following table:

# Voters	**4**	**1**	**9**	**8**	**5**
1st choice	A	B	C	A	C
2nd choice	B	A	D	D	D
3rd choice	D	D	A	B	B
4th choice	C	C	B	C	A

(a) Find the winner under the Borda count method.

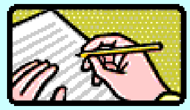
(b) Explain why this election illustrates a violation of the majority criterion.

(c) Explain why this election illustrates a violation of the Condorcet criterion.

CHECK

With Online Answer Sheet

WORK



Problem 24 on Page 34 of Text

An election with three candidates and 100 voters is to be determined using the Borda count method.

(a) What is the maximum number of points a candidate can receive?

(b) What is the minimum number of points a candidate can receive?

CHECK

With Online Answer Sheet

READ

Section 1.4 The Plurality with Elimination Method

Plurality-with-Elimination Method (AKA: Instant Runoff Voting or **Hare Method):** When no candidate gets the majority of the vote, rather than have a runoff election among the candidates with the most votes, time and money can be saved by having the initial vote on a preference ballot.

- **Round 1:** Count the first-place votes for each candidate, just as you would in the plurality method.
  - IF a candidate has a majority of first-place votes, that candidate is the winner.
  - IF NOT, then eliminate the candidate (or candidates if there is a tie) with the fewest first-place votes.
- **Round 2:** Cross off the names of the candidates eliminated from the preference schedule and recount the first-place votes. (Remember that when a candidate is eliminated from the preference schedule, in each column the candidates below it move up a spot.)
  - IF a candidate has the majority of first-place votes, declare that candidate the winner.
  - IF NOT, eliminate the candidate with the fewest first-place votes.
- **Round 3, 4, etc.** Repeat the process as needed until there is a candidate with the majority of first-place votes.

Using Plurality with Elimination Method on MAS Election

Apply the Plurality-with Elimination Method to the MAS election

Preference Schedule for MAS Election					
# of Voters	14	10	8	4	1
1^{st} Choice	**A**	**C**	**D**	**B**	**C**
2^{nd} Choice	**B**	**B**	**C**	**D**	**D**
3^{rd} Choice	**C**	**D**	**B**	**C**	**B**
4^{th} Choice	**D**	**A**	**A**	**A**	**A**

Round 1 Results:

Candidate	A	B	C	D
Number of 1^{st} place votes				

Round 2 Results:

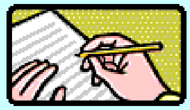
Candidate	A	B	C	D
Number of 1^{st} place votes				

Round 3 Results:

Candidate	A	B	C	D
Number of 1^{st} place votes				

And the winner is ____________________!!!

WORK



The XYZ Corporation is holding an election to choose a new Chairman of the Board. The candidates are Allen, Beckman, Cole, Dent, and Emery. The preference schedule for the election is given in the following table:

# voters	10	8	5	4	3
1st choice	A	D	B	C	E
2nd choice	C	C	C	B	A
3rd choice	B	B	D	D	C
4th choice	D	E	A	E	B
5th choice	E	A	E	A	D

(a) Find the winner of the election under the plurality-with-elimination method.

(b) There is a Condorcet candidate in this election. Find it.

(c) Explain why the plurality-with-elimination method violates the Condorcet criterion.

CHECK

With Online Answer Sheet

Monotonicity Criterion

Monotonicity Criterion: If choice X is a winner of an election and, in a reelection, the only changes in the ballots are changes that only favor X, then X should remain a winner in the election.

Weaknesses of the Plurality-with-Elimination Method: the Monotonicity Criterion and Condorcet Criterion can both be violated using this method, as the following example shows.

Example 1.10. Three cities, Athens (A), Babylon (B), and Carthage (C) are competing to hose the next Summer Olympic Games. The final decision is made by a secret vote of the 29 members of the Executive Council of the International Olympic Committee, and the winner is chosen using the plurality-with-elimination method.

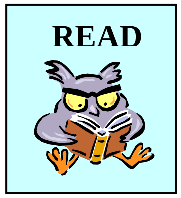
Two Days Before the Actual Election				
# voters	7	8	10	4
1^{st}	A	B	C	A
2^{nd}	B	C	A	C
3^{rd}	C	A	B	B

Using Plurality-with-Elimination Method, who wins this "straw" vote and with how many votes?

Actual Vote 2 Days Later			
#voters	7	8	14
1^{st}	A	B	C
2^{nd}	B	C	A
3^{rd}	C	A	B

The only difference is that the four who had voted (A, C, B) in the straw vote have now voted (C, A, B), switching only their first and second place votes and making a total

Using Plurality-with-Elimination Method to determine the winner of this vote. It should be C, right?



Section 1.5 Pairwise Comparison
(Copeland's Method)

Step 1: List all the pairs. Remember that the pair A-B would be the same comparison as B-A. For N candidates, there will be $\frac{(N-1)N}{2}$ pairwise comparisons.
Below is a quick reference for pairwise comparisons of 3, 4, or 5 candidates.

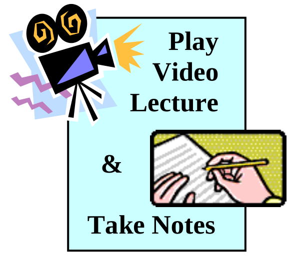
3 candidates ➔3 comparisons	4 candidates ➔6 comparisons	5 candidates ➔10 comparisons
A-B	A-B	A-B
A-C	A-C	A-C
B-C	A-D	A-D
	B-C	A-E
	B-D	B-C
	C-D	B-D
		B-E
		C-D
		C-E
		D-E

Step 2: For each pairwise comparison in your list, locate the pair in the first column of the preference schedule. Whichever of the two is listed higher in the column, "wins" the votes in that column.

Do this for each ballot column for the first pairwise comparison. Add up the votes for that each member of that pair. Whichever member of the pair has more votes after all the ballots are counted "wins". A "win" is worth 1 point in the final count. A "tie" is worth ½ point. A "loss" is worth 0 points.

Now do the same thing for each of the subsequent pairwise comparisons.

Step 3: Total the number of points collected this way by each candidate. The candidate with the most points wins.

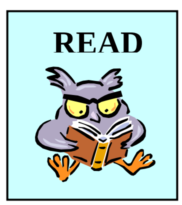


MAS Election with Pairwise Comparison Method

Example: MAS Election with Pairwise Comparison Method

#voters	14	10	8	4	1
1^{st}	A	C	D	B	C
2^{nd}	B	B	C	D	D
3^{rd}	C	D	B	C	B
4^{th}	D	A	A	A	A

A-B A-C A-D B-C B-D C-D



Independence-of-Irrelevant-Alternatives Criterion

Independence-of-Irrelevant-Alternatives Criterion: If choice X is a winner of an election and one (or more) of the other choices is disqualified and the ballots recounted, then X should still be a winner of the election.

Weaknesses of Pairwise Comparison Method

- can violate independence-of-Irrelevant-Alternatives Criterion
- can result with every candidate tied for first place

Example Violation of Independence-of-Alternatives Criterion

Example of Violation of Independence-of-Irrelevant-Alternatives Criterion

Preference Schedule for LA's Draft Choices							
#voters	2	6	4	1	1	4	4
1st	A	B	B	C	C	D	E
2nd	D	A	A	B	D	A	C
3rd	C	C	D	A	A	E	D
4th	B	D	E	D	B	C	B
5th	E	E	C	E	E	B	A

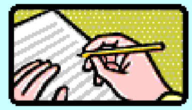
A-B A-C A-D A-E B-C B-D B-E C-D C-E D-E

What happens if Castillo (C) drops out of the Race? Since he did not win, will his dropping out affect the outcome?

Preference Schedule after Castillo Drops out							
#voters	2	6	4	1	1	4	4
1st							
2nd							
3rd							
4th							

A-B A-D A-E B-D B-E D-E

WORK



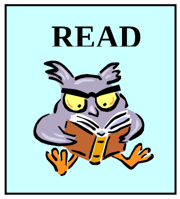
Problem 19 Page 33 Text

Find the winner of the election given by the following preference table under the method of pairwise comparisons.

# voters	8	7	6	2	1
1st choice	A	D	D	C	E
2nd choice	B	B	B	A	A
3rd choice	C	A	E	B	D
4th choice	D	C	C	D	B
5th choice	E	E	A	E	C

CHECK

With Online Answer Sheet



Section 1.6 Rankings

Simple Extended Rankings

Each of the methods of vote counting we have studied can be extended to ranking the candidates in 1^{st}, 2^{nd}, 3^{rd}, etc. place as well as simply determining a winner.

- Extended Plurality Method: rank by number of 1^{st} place votes received
- Extended Plurality-with-Elimination Method: rank in reverse order of elimination
  And remember that you don't choose 1^{st} place until majority is reached for some candidate
- Extended Borda Count: rank by number of Borda points awarded
- Extended Pairwise Comparison: rank by number of pairwise comparisons won

Recursive Extended Rankings

Step 1: use the regular method to pick 1^{st} place.

Step 2: eliminate the winner from the preference schedule and apply the same method again. The winner this time is awarded 2^{nd} place.

Step 3: eliminate the winner from the preference schedule and apply the same method again. The winner this time is awarded 3^{rd} place . . . etc.

End of Chapter 1 Checklist

Before beginning the next Part be sure you have done the following:

_____1. Worked through all the guided notes for
Part 8: The Mathematics of Voting

_____2. Worked through all the accompanying online video sessions for
Part 8: The Mathematics of Voting

_____ 3. Checked all the "WORK" pages with the online answer sheets for
Part 8: The Mathematics of Voting

_____4. Completed, checked and corrected all the homework problems from Chapter 1 in the text Pages 30 – 38 #1, 4, 6, 13, 19, 20, 24, 27, 34, 35, 36, 41, 44, 49, 51, 59

_____5. Completed and turned in the **online quiz** for Chapter 1
Part 8: The Mathematics of Voting

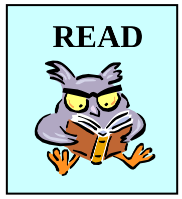
(Chapter 2 in textbook)

To the student:

- **This Part corresponds to Chapters 2 in the textbook**
- **First, read Chapter 2. Read for the BIG IDEAS and basic meaning. Don't worry about the details. We will do that in the guided notes.**
- **Then work through these guided notes**
- **Then do the homework assignment below**

  Homework: Pages 72 – 76 #1, 3, 4, 7, 12, 15, 23, 28, 45, 48 & 54

Associated Online Resources for Part 9 Guided Notes (GN)		
GN Page	**Type of Online Resource**	**Name of Online Resource**
208	**Video Lecture**	Weighted Voting System
209	Answer Sheet	Problem #2 page 72
210	**Video Lecture**	Problem #4 page 72
211	Answer Sheet	Problem #10 page 73
213	**Video Lecture**	Problem #12 page 73
215	**Video Lecture**	Example Banzhaff Power Index
216	Answer Sheet	Problem #16a page 74
218	**Video Lecture**	Problem #24 page 75
219	Answer Sheet	Problem #26 page 75
	Online Quiz	Online Quiz on Chapter 2



Chapter 2: Weighted Voting Systems: The Power Game

Weighted Voting System: Any formal voting arrangement in which the voters are not necessarily equal in terms of the number of votes they control. The voters may be corporations, organizations, states, countries rather than individuals.

Motion: A vote that has only two choices, usually thought of as "Yes-No" votes.

Players: who votes in a weighted voting system
N = number of players
$P_1, P_2, P_3, ..., P_N$ represent the players' names

Weight: How many votes a player controls.
$w_1, w_2, w_3, ..., w_N$ are the respective weights of the players $P_1, P_2, P_3, ..., P_N$

Quota: Minimum number of votes needed to pass a motion. Symbol is $\boldsymbol{q}$.
A quota can be something other than a strict majority of votes – it could take even more than a majority to pass a motion, depending on the rules & definitions in the situation.

If the quota does not come out to be a counting number (has fractional or decimal parts), BE SURE YOUR ROUND UP.

50% of total possible votes $\le \boldsymbol{q} \le$ 100% of total possible votes
Otherwise, the voting system is "not legal".

A quota of "at least 2/3" and a quota of "more than 2/3" are different.

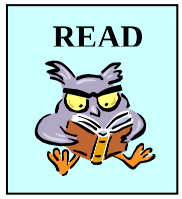
Notation for Weighted Voting System:

Weight of Player 1's Vote

Weight of Player 2's Vote, **Etc.**

quota amount

$$[\, q : w_1,\ w_2,\ w_3,\ ...,\ w_N \,]$$



Examples of Weighted Voting Systems

Example
There are weighted voting systems that are equivalent to 1 person-1 vote.

[11: 4, 4, 4, 4, 4] all five players have equal number of votes and this system works out exactly like a 1-person/1-vote, majority wins [5: 1, 1, 1, 1, 1]

Example
Equal number of votes is not the same as equal say in the outcome of the election. The number of votes a player controls can be very deceptive

[15: 5,4,3,2,1] works the same as [5: 1, 1, 1, 1, 1] since the quota is 15, the vote must be unanimous for the motion to pass.

This means that player P_5 can still stop the motion from passing, and therefore P_5's single vote can do everything that P_1's 5 votes can do.

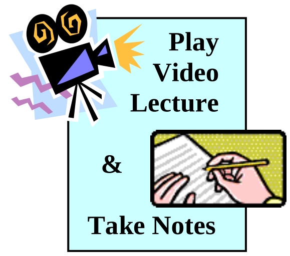
Dictator, Dummy, and Veto Power

Dictator: a player with weight ≥ the quota. In this case, the player can pass or defeat a motion by himself.

Dummy: a player whose votes have no power to change the outcome.

- When there is a dictator, all other players are dummies.
- Can also happen when no coalition needs this player's vote to control the outcome.

Veto Power: when a player has enough weight to prevent the rest of the players from passing a motion (all the others' weight together is not enough to pass a motion).



Weighted Voting Systems

Example: [11: 12, 5, 4]

P_1 is a ___________________________ because_______________________________

P_2 is a ___________________________ because ________________________________

P_3 is a ___________________________ because ________________________________

Example: [12: 9, 5, 4, 2]

P_1 is not a dictator, but does have veto power. Explain why:___________________

____

____

Problem 8, text page 72.

(a) [95: 95, 80, 10, 2]

Dictator?

Veto Power?

Dummies?

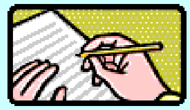
(c) [48: 32, 16, 8, 4, 2, 1]

Dictator?

Veto Power?

Dummies?

WORK



Problem #2

In a weighted voting system [31: 12, 8, 6, 5, 5, 5, 2] find:

(a) the total number of players

(b) the total number of votes

(c) the weight of P_3

(d) the minimum percentage of the votes needed to pass a motion
(remember to round up to the next whole percent)

CHECK

With Online Answer Sheet

Problem #6 from Page 72 in Text

A committee has 6 members $(P_1, P_2, P_3, P_4, P_5, P_6)$. In this committee P_1 has twice as many votes as P_2; P_2 *and* P_3 have the same number of votes, which is twice as many as P_4; P_4 has twice as many votes as P_5. P_5 *and* P_6 have the same number of votes. Describe the committee as a weighted voting system when the requirements to pass a motion are:

(a) a simple majority of the votes

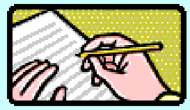
(b)at least three-fourths of the votes

(c) more than three-fourths of the votes

(d) at least two-thirds of the votes

(e) more than two-thirds of the votes

WORK



Problem #10 from Page 73 Text

In each of the following weighted voting systems, determine which players, if any, (i) are dictators; (ii) have veto power; (iii) are dummies.

(a) [27: 12, 10, 4, 2]

(b) [22: 10, 8, 7, 2, 1]

(c) [21: 23, 10, 5, 2]

(d) [15: 11, 5, 2, 1]

CHECK

With Online Answer Sheet

READ

Section 2.2 The Banzhaf Power Index

Coalition: any set of players that join forces to vote together.

Weight of Coalition: total weight of votes in coalition.

Winning Coalition: Coalition with enough votes to win.

Losing Coalition: Coalition that does not have enough votes to win

Grand Coalition: Coalition of all the players. A Grand Coalition is always a winning coalition in a "legal" voting system.

Critical Player: Player, who by deserting the coalition, can turn a winning coaling into a losing coalition. It is possible that a coalition have more than one critical player. Losing coalitions do not have critical players.

Notation: Express coalitions as sets of voters, using { } to enclose the voters who have formed a coalition to vote together on a motion.

Possible number of coalitions for a set with N elements: $2^N - 1$ because there are 2^N possible subsets for a set with N elements, but the empty set (always a subset) cannot be a coalition, so we subtract 1 to remove the empty set.

Problem 12 Page 73 in Text

Consider the weighted voting system [5: 3, 2, 1, 1].

(a) What is the weight of the coalition formed by P_1 and P_3?

(b) Which players are critical in the coalition $\{P_1, P_{2,,} P_3\}$?

(c) Which players are critical in the coalition $\{P_1, P_3, P_4\}$?

How many coalitions are possible?

(d) Write out all the coalitions and mark the **winning** coalitions.

READ

Banzhaf Power Index (1965)

Banzhaf Power Index: a player's power is proportional to the number of coalitions for which the player is critical. The more often the player is critical, the more power that player has.

Step 1: Make a list of all possible coalitions.

Step 2: Determine which are winning coalitions.

Step 3: In each winning coalition, circle the players that are critical players for that coalition.

Step 4: Count the total number of times player P_1 is a critical player. Call this number B.

Step 5: Count the total number of "critical player" occurrences in the list. Call this number T.

The Banzhaf Power Index of player P_1 is the ratio $\frac{B}{T}$

The Banzhaf Power Distribution is the list of the form:

P_1 : _____, P_2 : _______, P_3 : _______, ... , P_N : ________

where the blanks are the power indices of the respective players expressed either as ratios or as percent.

Example: Banzhaff Power Index

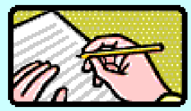
Example: In a family company, George I has 3 votes, George II, has 2 votes, and George III has 1 vote. It takes a majority to pass a motion.

(a) write the voting system in [:] notation

(b) How many possible coalitions exist?

(c) Find the Banzhaf Power Distribution for this voting system.

WORK

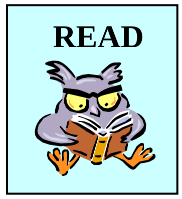


Problem #16a from Page 73 Text

Find the Banzhaf power distribution of the weighted voting system [9: 5, 5, 4, 2,1]

CHECK

With Online Answer Sheet



Section 2.4 The Shapley-Shubik Power Index (1954)

Sequential Coalition: Coalition taking into account the order the players joined the coalition.

Symbol $\langle P_1, P_2, P_3 \rangle$ is a sequential coalition of three players.

Factorial: the symbol 5! is read "five factorial" and is the product of

$5 \times 4 \times 3 \times 2 \times 1 = 120$

5! Represents the number of sequential coalitions that can be formed from 5 players.

N!: the number of sequential coalitions possible with N players.

Pivotal Player: The player who, when he joins the sequential coalition turns the coalition from a losing coalition to a winning coalition.

- Those who came before are not powerful enough
- Those who came after are not absolutely necessary for the coalition to win.

There is only one pivotal player per coalition.

Shapley-Shubic Power Index

Step 1: make a list of all sequential coalitions containing all N players (there should be N! of them)

Step 2: in each sequential coalition determine the pivotal player.

Step 3: count the total number of times P is pivotal. Call this value S.

Shapley-Shubic Power Index of $P = \dfrac{S}{N!}$

The value of the Shapley-Shubic Power Index can be the same as the value of the Banzhaf Power Index for the same weighted voting system, BUT USUALLY IT IS NOT.

Problem #24 from Page 74 Text

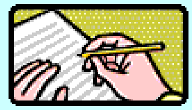
Consider the weighted voting system [8: 7, 6, 2]

(a) Write down all the sequential coalitions involving all three players.

(b) In each sequential coalition above, circle the pivotal player.

(c) Find the Shapley-Shubik power distribution for this weighted voting system.

WORK



Problem #26 from Page 74 Text

Find the Shapley-Shubik power distribution of the weighted voting system [60: 32, 31, 28, 21]

CHECK

With Online Answer Sheet

End of Chapter Checklist

Before beginning the next Part be sure you have done the following:

______1. Worked through all the guided notes for
Part 9: Weighted Voting Systems

______2. Worked through all the accompanying online video sessions for
Part 9: Weighted Voting Systems

______ 3. Checked all the "WORK" pages with the online answer sheets for
Part 9: Weighted Voting Systems

______4. Completed, checked and corrected all the homework problems from
Chapter 2 in the text Pages 72 – 76 #1, 3, 4, 7, 12, 15, 23, 28, 45, 48 & 54

______5. Completed and turned in the **<u>online quiz over Chapter 2</u>** for
Part 9: Weighted Voting Systems

Part 10: Fair Division

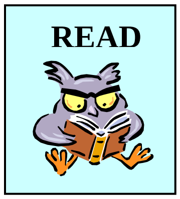
(Chapter 3 in textbook)

To the student:

- **This Part corresponds to Chapters 3 in the textbook**
- **First, read Chapter 3. Read for the BIG IDEAS and basic meaning. Don't worry about the details. We will do that in the guided notes.**
- **Then work through these guided notes**
- **Then do the homework assignment below**

  Homework: Pages 111 – 122 # 1, 4, 5, 11, 21, 30, 27, 31, 33, 40, 41, 49, 59, 64, 67, 75

Associated Online Resources for Part 10 Guided Notes (GN)		
GN Page	**Type of Online Resource**	**Name of Online Resource**
224	**Video Lecture**	Fair Share Example
226	**Video Lecture**	Two Person Divider-Chooser
229-230	**Video Lecture**	Example 3.4 from Text
231	Answer Sheet	Page 116 Problem 14
233	Answer Sheet	Example 3.6
235	**Video Lecture**	Lone Chooser Example
236-238	**Video Lecture**	Example 3.7 Lone Chooser Method
239-240	Answer Sheet	Problem 26 Page 120 Text
244-246	**Video Lecture**	Example 3.8 from Text p. 101-104
247	Answer Sheet	Problem 34 p. 123
248	**Video Lecture**	Problem 36 page 123 text
249-250	**Video Lecture**	The Method of Sealed Bids
251	Answer Sheet	Example 3.10
252	Answer Sheet	Problem 44 from p. 124 text
257	**Video Lecture**	Problem 52 from p. 127 text
258	Answer Sheet	**Problem 50 p. 127 text**
	Online Quiz	**Online Quiz Chapter 3**
Exam 3 over Parts 8 -10: Chapters 1 – 3		



Chapter 3: Fair Division

Overview: In this chapter we will study two categories of Fair Division Games

- **Continuous: things that can be "cut" anywhere like food, land, cake, pizza, fabric, etc. Methods for dividing continuous items are:**
  - Divider-Chooser Method
  - Lone Divider Method
  - Lone Chooser Method
  - Last Diminisher Method
- **Discrete: things that can't be "cut" just anywhere like paintings, houses, jewelry, cars, etc. Methods for dividing discrete items are:**
  - Sealed Bids Method
  - Method of Markers

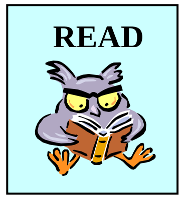
Components of Fair Division Games:

- **Goods** to be divided: anything with potential value – the "Booty". The symbol for set of all the goods to be divided ***S.*** Capital ***S*** is the whole set and small ***s*** with subscript indicates a particular share (a subset) of ***S***.

- **Players**: those entitled to a share of the goods – may be individuals, corporations, countries, etc. The symbols for the players are : $P_1, P_{2,} P_3, ... P_N$ where there are N players in the game.

- **Value Systems**: each player has own value system – the ability to assign value to S or any subset of S. Be careful: each player has his or her **<u>own value system</u>** and you have to deal with each one at each stage of the division.

Goal of a Fair Division Game: to divide S into one-share-for-each-player so that each player gets a "fair share" **<u>according to that player's value system</u>**.



Fair Division Methods (also called Protocols or Schemes) are based on:

- **A set of rules for division** (each method has different rules for dividing S) the goal of which is that after a finite number of moves, the game ends with Fair Division of S.

- **A set of assumptions about the players:** The games we will study have these assumptions:

  - **Cooperation:** each player is a willing participant and agrees to the terms of the game
  - **Rationality:** players act rationally and their value systems conform to the rules of arithmetic
  - **Privacy:** no player knows any other player's value system or how another player is going to move.
  - **Symmetry:** equal rights in sharing S ➔ each player is entitled to a proportional share of S.

Guarantee: When these four assumptions are satisfied, A Fair Division Method guarantees each player the opportunity to get a fair share of S. ➔ There is a strategy available to each player, that if followed correctly, guarantees he or she will receive a fair share.

Fair Share: (sometimes called a "proportional fair share") Given a share s of S and a player P of the game, "share ***s*** is a fair share to player P, **IF**, in the opinion of P, ***s*** is worth at least $\frac{1}{N}$ of the total value of ***S***."

Best Share: the share the person values most highly.

"Fair Share" is relative

- What is a fair share for 1 player may or may not be a fair share for another player
- Fair share does not mean every player gets exactly the same thing

Fair share is not necessarily the same thing as getting the "Best Share" the share you would prefer.

Fair Share Example

Example: 4 players and 4 shares.

Player Paul thinks
- s_1 is worth 15%
- s_2 is worth 35%
- s_3 is worth 20%
- s_4 is worth 30%

In this game Paul is entitled to _______ of the goods.

He would prefer share ______ because he values it most.

But _____ would also be a fair share for Paul because it is worth at least 25% of the total value to him.

Player Tanya thinks
- s_1 is worth 50%
- s_2 is worth 25%
- s_3 is worth 10%
- s_4 is worth 15%

In this game Tanya is entitled to _______ of the goods.

She would prefer share ______ because she values it most.

But _____ would also be a fair share for Tanya because it is worth at least 25% of the total value to her.

Tim and Tom both value the goods the same way. They think
- s_1 is worth 25%
- s_2 is worth 20%
- s_3 is worth 40%
- s_4 is worth 15%

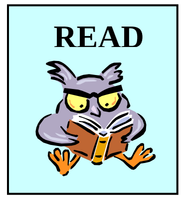
In this game Tim and Tom are each entitled to _______ of the goods.

Each one would prefer share ______ because he values it most.

But _____ would also be a fair share for either Tim or Tom.

A fair division of these goods would be:

Paul	Tanya	Tim	Tom
S2 or S4	S1 or S2	S1 or S3	S1 or S3



Two Players: The Divider-Chooser Method

Divider-Chooser is a method for dividing **continuous** “Booty”.

- “You cut, I choose.”
- The divider cuts so that either piece is acceptable
- The chooser chooses the piece he likes best.
- Method guarantees each player a strategy to get what he or she perceives to be at least 50% of the value of the goods.

Strategy – Reasoning.

- The divider’s strategy is to divide the goods so that either part of the division is of equal value to the divider. So no matter which part the divider gets, he is guaranteed to receive ½ of the value of the goods, as he perceives the goods’ values.

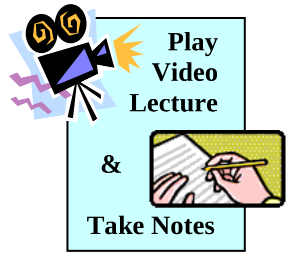
- The chooser gets the choice of the two pieces, so either the pieces are exactly equal in value in the chooser’s value system, or one is worth more than ½ the value in the chooser’s value system. The chooser’s strategy is to take the piece that the chooser ***perceives as ½ the value of the goods or more***.

- It is possible that the chooser gets more perceived value than the divider, but each is guaranteed the opportunity to get a share that is worth at least ½ the total value of the goods according to that player’s value system.

- Because the chooser may have the opportunity to choose a share worth more than ½ by the chooser’s value system, it is preferable to be the chooser.

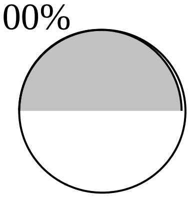
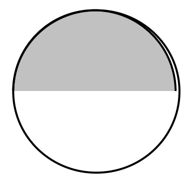
- Therefore the roles of divider and chooser should be assigned at random – say with the flip of a coin.

But remember that by the rules of the game, a fair share is ***at least*** $\frac{1}{N}$ of the perceived value of the goods, and both divider and chooser will receive ***at least*** that much.

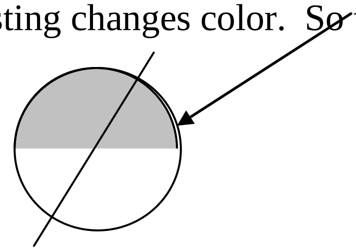


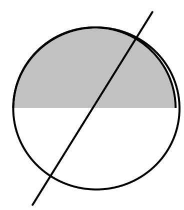
Two Person Divider-Chooser Example

Example 3.1 Suppose we have a cake that is ½ strawberry frosted and ½ chocolate frosted. To Damien the chocolate and strawberry are of equal value. But Cleo is allergic to chocolate and cannot eat it. So she values the chocolate at 0 and the strawberry at 100%

Damien sees value:  Cleo sees value: 

Suppose that Damien (by assumption not knowing how Cleo values the parts of the cake) divides the cake as follows. The cut is a 60° angle to the line where the frosting changes color. So the chocolate wedge formed here is 1/6 of the cake.





(a) How does Damien see the values of the two pieces?

(b) How does Cleo see the value of the two pieces?

Since Damien cut, Cleo gets to choose.

(c) Which piece should Cleo choose?

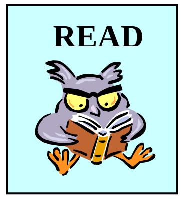
(d) How much value does Cleo place on that piece?

(e) Which piece does Damien get?

(f) How much value does Damien place on his piece?

(g) Is this "fair division" according to the rules of the game?

(h) How is it possible that the two percentages (His value + Her Value) add up to more than 100%?



The Lone-Divider Method

Lone-Divider Method: an extension of the Divider-Chooser Method that can be used with more than 2 players.

Here are the Steps for the Lone-Divider Method with 3 players:

- **Begin:** by random draw, players decide who is the lone-divider and who are the choosers

- **Step 1:** Divider D divides the cake into 3 pieces. Since D does not know which piece he will get ➔ he divides the cake into 3 pieces that are of equal value to him.

- **Step 2:** Bidding: Each chooser writes down which of the 3 pieces are worth 1/3 or more by his/her value system. They do not confer. They write their bids independently and secretly. Each may bid on 1, 2, or 3 of the shares.

It must be possible for each chooser to find at least 1 share that is worth 1/3 or more to the chooser. Why?

If all three pieces are less than 1/3, the whole thing will not add up to 1 whole – which it must since it is the whole thing.

- **Step 3:** Distribution. The bids are sorted. The shares are sorted. The shares that nobody bid on are placed in one group designated "U" for unwanted. The shares that somebody bid on are put in another group designated group "C" for Chosen.

(continues on next page)

One of two things must occur in groups U and C.

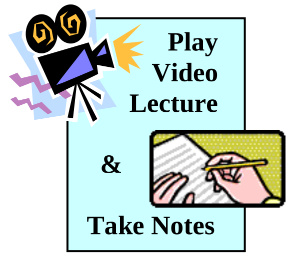
- Either there are 2 or more shares in the C group
- Or there is only1 share in C.

Case 1: if there are 2 shares in group C, then the divider gets the share in group U and the choosers receive one of the shares they bid on.

Case 2: if there is only 1 share in group C, the divider gets one of the shares in group U. There is a “standoff” between the other players. Put the other share from U together with the share in C, to form a new set of goods called B. Then use the divider-chooser method for the two choosers to divide B (the new group of goods) fairly.

Note: it is possible that each of the 3 players would rather have a different piece, BUT, the piece they got they valued as at least 1/3 of the set S. It is permissible for players to trade pieces after the assignments are made, if they so desire. But logically, they will only do so if the trade does not reduce the value of either persons share.

It is also possible that there is more than one way to make a “fair division” in a particular case.



Example 3.4 from Text

(a) How much is a Fair Share?

(b) How will the Choosers Bid?

	S_1	S_2	S_3
D	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
C_1	20%	30%	50%
C_2	10%	20%	70%

(c) What happens next?

(d) How much is set B worth to C_1?

(e) How much is set B worth to C_2?

(d) By the divider-chooser method, how much value is C_1 guaranteed to end up with in this situation?

(e) By the divider-chooser method, how much value is C_2 guaranteed to end up with in this situation?

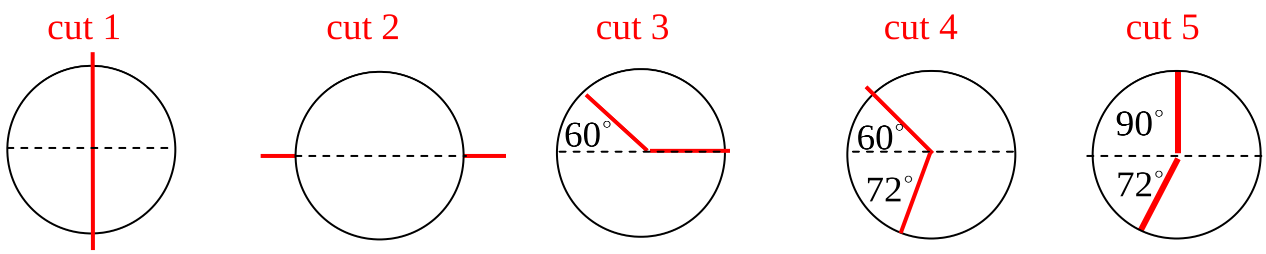
(e) Is this a fair division? Why or why not?

(continues)

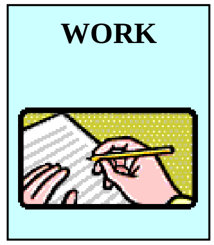
Problem 14 from Page 114 Text

Raul and Karli want to divide a chocolate-strawberry mouse cake. Raul values chocolate three times as much as he values strawberry. Karli values chocolate twice as much as she values strawberry.

(a) If Raul is the divider, which of the following cuts are consistent with Raul's value system?



(b) For each of the cuts consistent with Raul's value system, indicate which of the pieces is Karli's best choice.



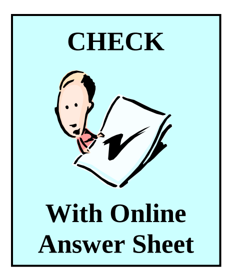
Problem 18 Page 115

Susan and Veronica want to divide an orange-pineapple cake using the divider-chooser method. Susan values orange four times as much as she values pineapple. Veronica is the divider and cuts the cake as shown in the following figure.

(a) What percent of the value of the cake is the pineapple half in <u>Veronica's</u> eyes?

(b) What percent of the value of the cake is each piece in picture (ii) in <u>Susan's</u> eyes?

(c) Describe the final fair division of the cake.



READ

Lone-Divider Method Extended to 4 Players

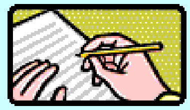
- Random selection of 1 divider. Everyone else is a chooser.
- Divider divides S into 4 shares.
- Each of the choosers makes a bid independently writing down which pieces are “fair shares” to that player. Note, that means they value the share $\geq \frac{1}{N}$ of S.

Case 1: From the bids, there is a way to give every chooser a fair share. Then the divider gets the last share.

Case 2: There are 2 or more bidding only on a single share (a standoff), or 3 bidding on only 2 shares, . . . or K choosers bidding on less than K shares. This is called a “standoff”.

- Set aside all the shares that are in the standoff.
- Separate the players involved in the standoff from the rest of the players
- Each of the remaining players (those not in the standoff) can be assigned a fair share from the remaining shares. They are done.
- All remaining shares are recombined to form a neew set S.
- The process starts over again for those involved in the standoff.

WORK



Example 3.6

	s_1	s_2	s_3	s_4
D	25%	25%	25%	25%
C_1	20%	20%	20%	40%
C_2	15%	35%	30%	20%
C_3	22%	23%	20%	35%

(a) How much is a Fair Share worth?

(b) Who bids on what?

(c) Is there a standoff?

(d) What are sets C and U?

(e) What do you do next?

(f) B's value to C_1 = _______
B's value to C_3 = _______

(g) How is the game finished in this case?

CHECK

With Online Answer Sheet

READ

Section 3.4 The Lone-Chooser Method

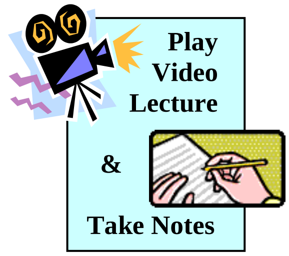
Lone Chooser Method with 3 Players

- **Start:** 1 chooser, 2 dividers chosen by random draw.

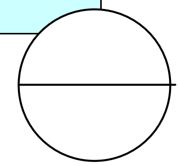
- **Step 1 The Division:** D_1 and D_2 follow Divider-Chooser Method to create 2 fair shares by their values.

- **Step 2 Subdivision:** Each divider divides his share into 3 subshares, all of equal value to the divider by his value system.
  D_1 creates 3 subshares: S_{1a}, S_{1b}, S_{1c} (each of these is of equal value to D_1)
  D_2 creates 3 subshares: S_{2a}, S_{2b}, S_{2c} (each of these is of equal value to D_2)

- **Step 3 Selection:** Chooser now chooses 1 piece from D_1 collection and 1 piece from D_2 collection. These two subshares make up C's final share. D_1 keeps his other two subshares and D_2 keeps his other two subshares.

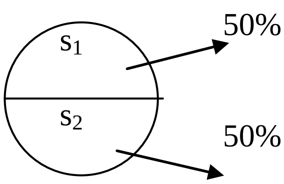


Lone Chooser Example

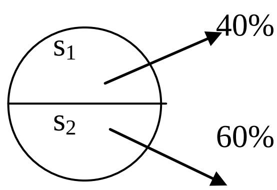


D_1 is randomly selected as the Divider and makes this division of the goods:

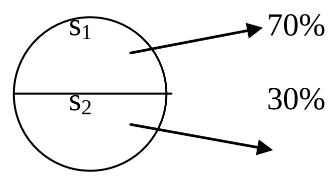
D_1's View of Values



D_2's View of Values



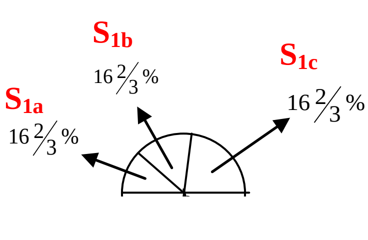
C's View of Values



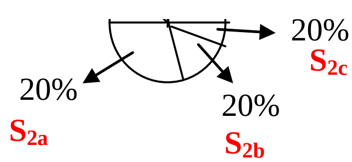
D_2 chooses ______ which leaves D_1 with _______. C is watching.

Then D_1 and D_2 each divide their share into 3 of ***what they consider equal value pieces***

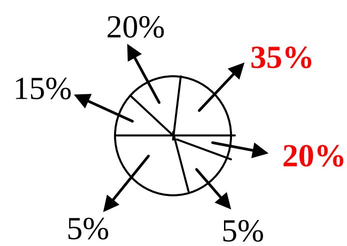
D_1's Division of s_1
in D1's view



D_2's division of s_2
in D2's view



C : the whole thing
in C's view



At this point, C chooses _____ from S_1 and ____ from S_2 for total value of ____.

Leaving D_1 with ______ and _______ for a total value of ______ (In D_1's system)

And Leaving D_2 with _____and ____ for a total value of _____(In D_2's system)

Is this a fair division?

Example 3.7 Lone Chooser Method

A total cake, valued by each player at \$27.00 is ½ orange and ½ pineapple. The three players are Dave, Cher and Dinah.

The three players draw straws for their roles: Cher gets the longest straw and becomes the chooser, C. Dave and Dinah get the two smaller straws, so they become the dividers, D_1 and D_2 respectively.

Dave values orange and pineapple equally.

(So orange ½ is worth \$____ and the pineapple ½ is worth \$______ to Dave.)

Cher values pineapple twice as much as orange.

(So orange ½ is worth \$____ and the pineapple ½ is worth \$______ to Cher.)

Dinah hates pineapple.

(So orange ½ is worth \$____ and the pineapple ½ is worth \$______ to Dinah.)

(continues)

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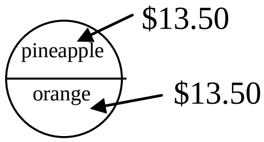
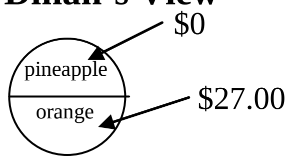
Step 1: Each Player's View of the cake at the beginning.

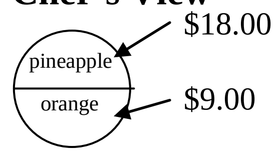
Daves' View



Dinah's View

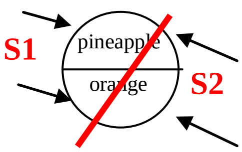


Cher's View



Step 2: Dave, D1, cuts the cake into 2 pieces with each piece worth 50% to him.
The small wedge is a 60° angle, (1/3 of the half or 1/6 of the whole cake).
Write in the value in dollars each would assign each piece in their view:

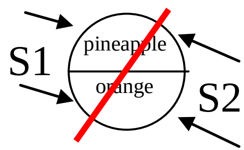
Daves' View



S1 is worth ______ to Dave

S2 is worth ______ to Dave

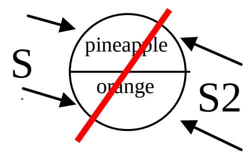
Dinah's View



S1 is worth ______ to Dinah

S2 is worth ______ to Dinah

Cher's View

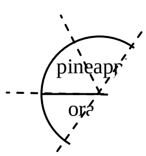


S1 is worth ______ to Cher

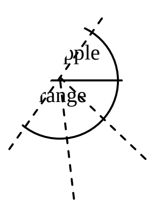
S2 is worth ______ to Cher

Step 3: Dinah chooses S2 because it has more orange and less pineapple.
This leave Dave with S1. **D**ave divides his piece into what he considers 3 equal shares. Since he values each flavor equally, he divides into equal thirds as shown below.
Dinah divides her piece into what she considers 3 equal shares, so that no matter which share she gets , she gets the same amount of orange because that is what she values most. Remember the Pineapple has no value to her.
Label all the remaining pieces with their dollar value in each person's view.

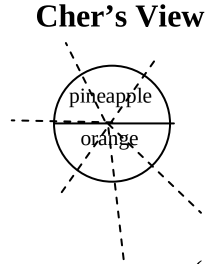
Daves' View



Dinah's View



Cher's View



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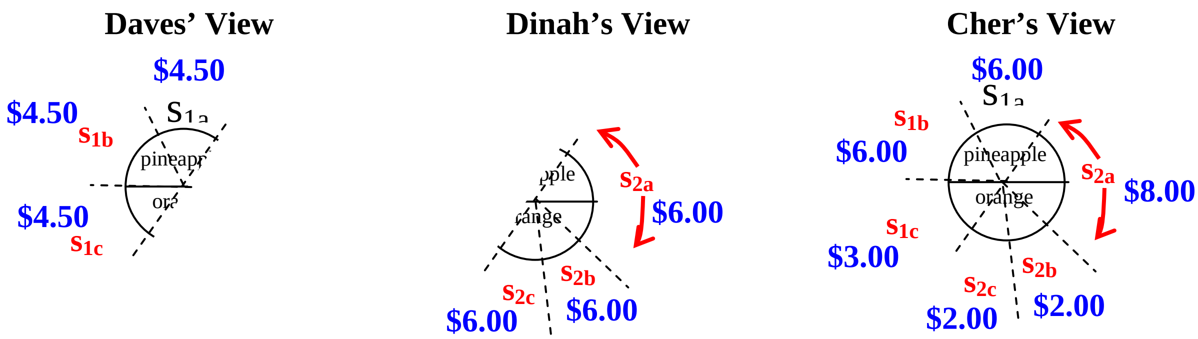
Step 4: Since Cher is the lone chooser, she chooses 1 piece from Dave's 3 subshares and 1 piece from Dinah's 3 subshares.



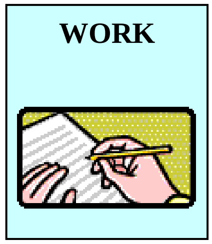
Which subshares does Cher pick?

What is the total value of Cher's 2 subshares in Cher's view?

What is the total value of Dinah's remaining 2 subshares in Dinah's view?

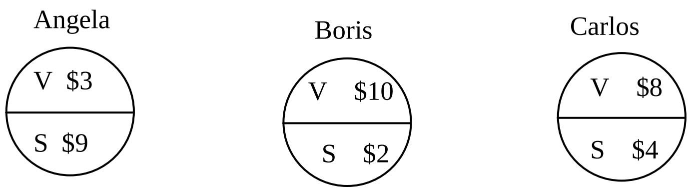
What is the total value of Dave's 2 remaining subshares in Dave's view?

Is this a fair division of the original cake?   Why or why not?



Problem 33 Page 117 Text

Angela, Boris, and Carlos decide to divide a $12 vanilla-strawberry cake using the lone-chooser method. The dollar amounts of the cake in each players eyes are given in the following figure:



Suppose that Carlos and Angela are the dividers and Boris is the chooser. In the first division, Carlos cuts the cake vertically through the center and Angela picks the right half.

(a) Draw this division and label Angela's and Carlos's halves.

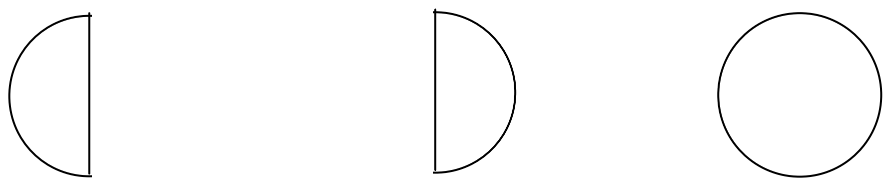
(b) Draw a possible second division that Carlos might make of the left half of the cake.

(c) Draw a possible second division that Angela might make of the right half of the cake.

(continues on next page)

(d) Summarize the values of the subsections in each person's view.

Carlos (D1) Angela (D2) Boris (C)



(e) Based on the second divisions you gave in (a) and (b) above, describe the a possible final fair division of the cake.

Boris gets subsections ____________ and ________________.

Angela gets subsections ___________ and _______________.

Carlos gets subsections ___________ and _______________.

(f) For the final fair division you described in (c), find the dollar value of each share in the eyes of the player receiving it.

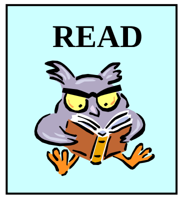
Boris values his share at _________________.

Angela values her share at _______________.

Carlos values his share at ________________.

CHECK

With Online Answer Sheet



Lone Chooser for N Players

Start: Randomly select who will be the lone chooser and the rest will be dividers. So the players are C, D_1, D_2, D_3, D_4, . . . D_{N-1}

This is an "inductive" strategy. It says if we can do it for 3 players, then we can do it for 4 players. If we can do it with 4 players, we can do it with 5 players . . .

So we know the following steps are true for four players, because we already did it with 3 players, so we know it is possible for the 3 Dividers to divide the beginning set S of goods into 3 fair shares.

Then if we know it works for 4 players, we can do 5 players, 4 of whom are Dividers. And so on . . .

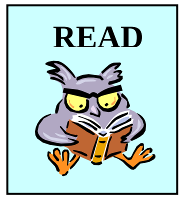
Step 1: Dividers divide S into fair shares among themselves as if C did not exist. Each should get a share that has a value of $\frac{1}{N-1}$ to that player, within that player's value system.

Step 2: Each divider divides his share into N subshares that he finds equally desireable (have same value to this divider). There are now a total of N(N-1) sub shares in the set S.

Step 3. Chooser choses 1 subshare from each of the dividers' portions $= \frac{(N-1)\ subshares}{(N-1)(N)\, total\ \ shares}$ which simplifies to $\frac{1}{N}$ the chooser's fair share.

Each of the Dividers keeps his remaining subshares. Each divider now also has

$= \frac{(N-1)\ subshares}{(N-1)(N)\, total\ \ shares}$ which is also $\frac{1}{N}$, a fair share each.



The Last Diminisher Method

Basic Game Plan:

The game starts with N players. In each round we find 1 player's fair share, and that player and that player's fair share will drop out of the game.

We do the next round by repeating the steps with the remaining N-1 players. At the end of the round, another player and his/her fair share drop out of the game.

We do the next round by repeating the steps with the remaining N-2 players (remember, 2 players have now dropped out).

We continue until all N players have determined their fair share and dropped out.

Game Description:

At any point in the game, S is in exactly 2 shares, a claimed piece called "C" and the rest which we will call "R".

At any point in the game, there are also 2 categories of players. There is one claimant who is claiming share C. Everyone else is a non-claimant.

As the game proceeds, each non-claimant has the opportunity to become a claimant by "trimming" the C piece or remaining a non-claimant.

So as the game progresses,

- the C-piece changes
- the R-piece changes
- who is the claimant changes

All of these factors together keep the players honest and the resulting shares "fair".

READ

The Last Diminisher Method – Game Steps

Start: randomly assign player order P_1, P_2, P_3, . . . P_N. (For instance, each player could draw a number from a hat, in which there are the numbers 1, 2, 3, ..., N)

Round 1: P1 starts as the claimant, cutting a share that he/she believes has value exactly $\frac{1}{N}$ of S. Strategy: If P_1 claim is excessive, P_1 will very likely loose the claim. If P1's claim is too small, P1 will be stuck with it just the way it is.

After P1 has "cut" a share, P2 has a choice. P2 can "trim" P1's share smaller and take it from P1 (the trimmed part goes back with the rest of the unclaimed goods) OR P2 can pass, if he/she agrees that what P1 has done is a fair share. During the round, the turns go in order down the ranks of the players, with each subsequent player either "passing" or trimming the claim and taking it for his own.

After the Nth player has made his choice, the round is over. Whoever currently is claimant gets to keep whatever is currently designated as the claim. The claimant and the claim leave the game. There are now N-1 players and the new set S of goods is the Rest "R" left over at the end of round 1. It is time for round 2.

Round 2: The whole process repeats, still going in the predetermined order of players. At the end of each round, since every player has had a chance to decide if the C piece claimed is too Big, they should be satisfied that the player who is now leaving the game with the C piece is leaving with $\leq \frac{1}{N}$ of the original whole. Each remaining player stills has a chance to get a piece that is $\geq \frac{1}{N}$ of the original whole ***according to that player's value system.***

Last Round: is the basic Divider-Chooser Method between the last two players remaining in the game.

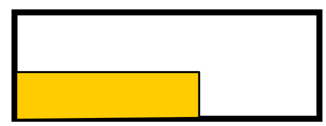
Example 3.8 from Text p. 100-103

5 sailors are marooned on an island, with no hope of escape. They decide to lay claim to the island with private ownership, deciding who gets what piece of land using the last Diminisher Method. Let's talk our way through the story. To simplify the drawings, we will use a rectangle to represent the island.

ROUND 1

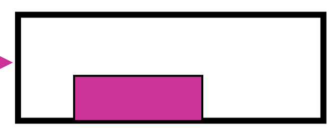
Randomly choose order in which the players will make their claims:
P1, **P2**, **P3**, **P4**, **P5**

P1 marks a portion of the land P1 would accept as a "fair share" 20% value. See the area shaded below.

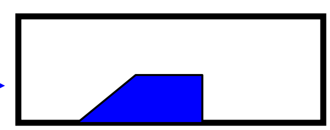


P2 "passes" accepting that P1's choice is OK. It has less than 20% of the value according to P2's system and P2 is willing to let P1 have it.

P3 "plays" modifying P1's choice making it smaller. This is now P3's claim. P3 would accept this as a "fair share".

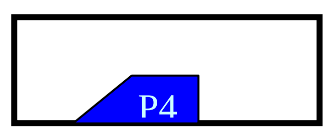


P4 "plays" modifying P3's choice making it even smaller. This becomes P4's claim. P4 would accept this as a "fair share".



P5 "passes" because he thinks the portion P4 has claimed is less than 20% of the total value.

This ends ROUND 1 because each player has had an opportunity to choose or diminish. Since **P4** is the **last diminisher** and gets to keep the portion he claimed.
P1, **P2**, **P3**, **P5** go on to **ROUND 2**



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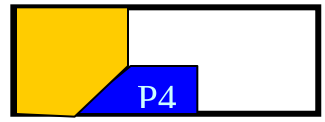
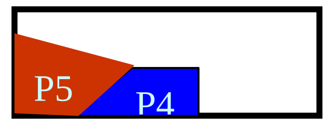
ROUND 2

P1 is the first player in round 2 (following the original order with P4 no longer playing). So P1 marks off what he would consider a "fair share" (or 25% of the remaining value since there are only 4 players left) of the rest of the island.



P2 and P3 both "pass" considering the portion P1 has marked to be worth less than 25% of the remaining island.

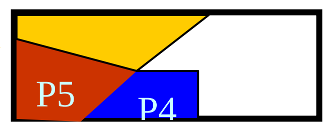
P5 "plays" because he believes the part marked by P1 is worth more than 25%, so P5 modifies P1's section to make it what P5 would consider a "fair share" or 25% of the remaining island. Since he is the **last diminisher, P5** gets to keep this claim.



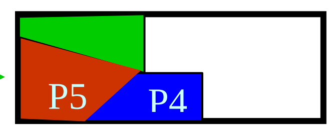
P1, P2, and P3 go on to the next round.

ROUND 3

P1 is once again the first player in round 3. So P1 marks another "fair share" off from the remaining parts of the island. (1/3 of the value of the unclaimed portions of the island)

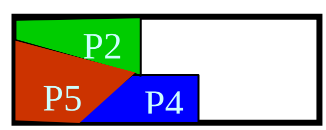


P2 "plays" because he believes the portion P1 has marked is worth more than of the remainder of the island. So P2 reduces the size of



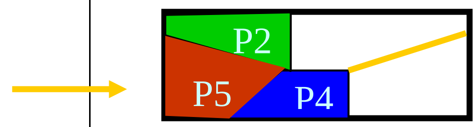
P3 "passes" believing that the section P2 has marked off is worth less than 1/3 of the value of the unclaimed portions of the island.

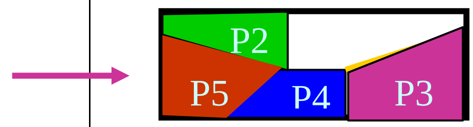
So P2 is the **last diminisher** and get to keep his claim.

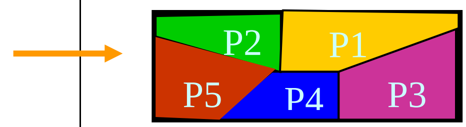


P1 and P3 go on to the next round.

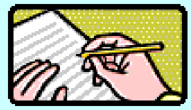
ROUND 4
Since there are only two players left, **P1** and **P3**, use the divider-chooser method to split the remaining unclaimed portion of the island. **P1** is the divider and makes the following division. **P3** is the chooser and chooses the lower section. And so **P1** gets the last remaining unclaimed section.







WORK



Problem

A cake is to be divided among four players (P1, P2, P3, and P4) using the last-diminisher method.

The players play in a fixed order, with P1 first, P2 second, etc.

In round 1, P1 cuts a piece ***s***, P2 and P3 pass and P4 diminishes it.

(a) Is it possible for P2 to end up with any part of s in his final share?

(b) Which player gets a piece at the end of round 1?

(c) Which player cuts the piece at the beginning of round 2?

(d) Who is the last player who has an opportunity to diminish the piece in round 2?

CHECK

With Online Answer Sheet

Problem 46 page 119 of Text

Problem 36 page 123: A cake is to be divided among six players (P1, P2, P3, P4, 5, P6) using the last-diminisher method. The players play in a fixed order, with P1 first, P2 second, etc.

In round 1, P1 cuts a piece, and P2, P3, and P6 are the only diminishers.
In round 2 there are no diminishers.
In round 3, after the first cut, each successive player is a diminisher.

(a) Which player gets the piece at the end of round 1?

(b) Which player cuts the piece at the beginning of round 2?

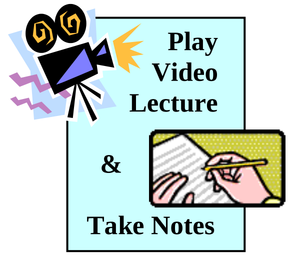
(c) Who is the last player with an opportunity to diminish the piece in round 2?

(d) Which player gets the piece at the end of round 2?

(e) Which player cuts the piece at the beginning of round 3?

(f) Which player gets the piece at the end of round 3?

(g) Who is the last player with an opportunity to diminish the piece in round 4?



The Method of Sealed Bids – Discrete Fair Division

In her will, Grandma leaves just three valuable items -- a house, a Rolls Royce, and a Picasso painting -- to her four grandchildren: Art, Betty, Carla, and Dave. She stipulates that the items may not be sold to outsiders and must be divided fairly in equal shares among them.

Step 1: Bidding: each player is asked to make a bid for the items in the estate, giving his or her honest assessment of the dollar value of each item. They write down their bids independently and no player knows what the other bids.

The Bids Summarized

	Art	Betty	Carla	Dave
House	220,000	250,000	211,000	198,000
Rolls Royce	40,000	30,000	47,000	52,000
Picasso	280,000	240,000	234,000	190,000

Step 2: **Allocation:** Each item goes to the highest bidder for that item.

- House:
- Rolls Royce:
- Picasso:
- Carla:

(continues)

Step 3: **Payments:** Calculate how much each player believes his fair share is worth. Do this by adding all of the players' bids and dividing by the number of players.

	Art	Betty	Carla	Dave
House	220,000	250,000	211,000	198,000
Rolls Royce	40,000	30,000	47,000	52,000
Picasso	280,000	240,000	234,000	190,000
Total				
Fair Share				

Now each player pays the estate the difference between what they received and their "fair share".

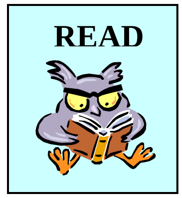
	Art	Betty	Carla	Dave
Item Value	280,000	250,000	0	52,000
Fair Share	135,000	130,000	123,000	110,000
Owes Estate				

Step 4: Dividing the Surplus:

At which point, each player has a "fair share" and gets a bonus of _______cash.

Final Division of all goods:

Art Betty Carla Dave

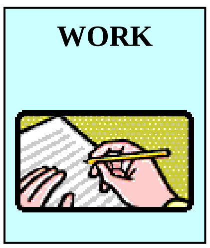


Sealed Bids works well as long as:

- Each player must have enough money to play the game – ie, enough money to pay the difference between their bid and the amount of a fair-share.
- Each player must be willing to accept money as a substitute for any item. It won't work if a player considers an item "priceless".

Sealed Bids work especially well to divide 1 discrete item between two people.

Example 3.10



Example 3.10 Al and Betty are getting a divorce. The only common property of value is their house. They decide to divide the house using sealed bids. Betty bids $142,000 for the house and Al bids $130,000 for the house.

Who gets the house? ______________________

What is the amount of a fair share? ___________________________________

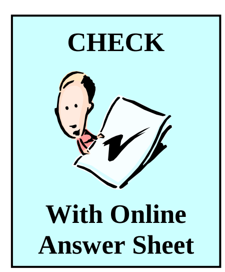
How much does Al get in the first part of the division? _________________

How much surplus is there?______________________________

The final settlement is:

Al   Betty

Is there another way to arrive at precisely this solution for the two-player case?



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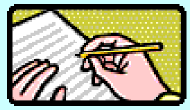


Problem 56 from Page 121 Text

Three players (A, B, and C) wish to divide up five items using the method of sealed bids. Their bids on each of the items are given in the following table.

	A	B	C
Item 1	$14,000	$12,000	$22,000
Item 2	24,000	15,000	33,000
Item 3	16,000	18,000	14,000
Item 4	16,000	16,000	18,000
Item 5	18,000	24,000	20,000

Describe the final outcome of this fair-division problem.

CHECK

With Online Answer Sheet

READ

The Method of Markers

Advantage: none of the players have to put up any cash as part of the division.

Required Conditions:
- There must be many more items to divide than there are players.
- The items need to be of similar value or it becomes too difficult for the players to decide where to place their markers.

Outcome of the Game: Each player comes away with one of the "strings" of items he or she bid on.

Steps for Method of Markers

Step 1: (Bidding) Each player independently divides the array into N fair shares by placing N-1 markers. The markers separate each fair share from the next.

Step 2: (Allocations) Scan the array from left to right until the first of the First Markers is located. The player owning that marker gets to keep his first segment and all the rest of that player's markers are removed. In case of a tie, break the tie randomly. We continue moving from left to right, looking for the first of the SECOND MARKERS. The player owning it gets to keep her second segment. Continue this process until each player has received one of the segments.

Step 3: (Leftovers). The leftover items can be divided among the players by some form of lottery or if there are many leftovers, the method of markers may be repeated.

READ

Example 3.11 from Text

Four children, Alice, Bianca, Carla, and Dana (A, B, C, D,) are to divide 20 pieces of candy

Start: Line up items to be divided in an array—a linear sequence that cannot be altered during the division. Think of this as a long "string" of objects.

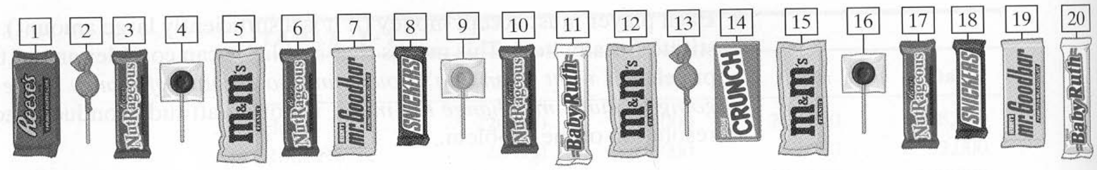


FIGURE 3-17
The original array.

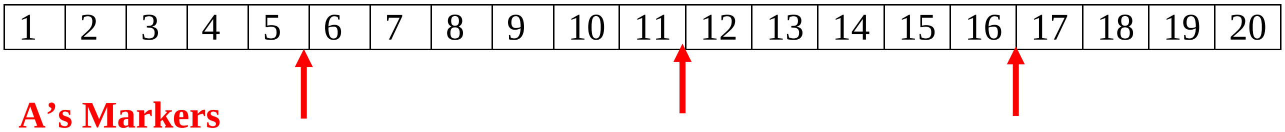
Step 1: **Bidding:** Each player creates a bid in which he "cuts" the string of objects into what he believes are "N" fair shares – one for each of the players in the game. In this case, there are 4 players, so each player writes down how they would "cut" the string by placing 3 (N-1) markers. No player should see any other player's markers before laying down his own markers. The method guarantees that each player ends up with one of his or her bid segments of the "string" of items.

For instance, each player could mark a numbered diagram like this with the player's markers, representing how that person would divide the string into N sections of equal value to that player. Notice that it requires (N-1) markers to divide the string of items into N sections.

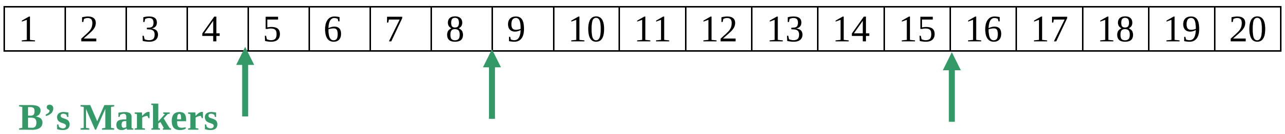
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

(continues)

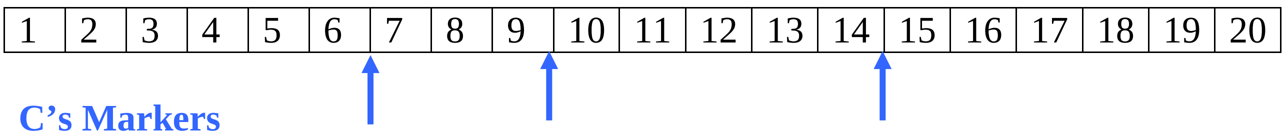
Suppose each player submits the following bids.



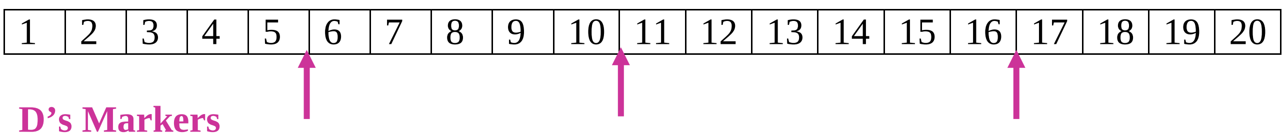
A's Markers



B's Markers

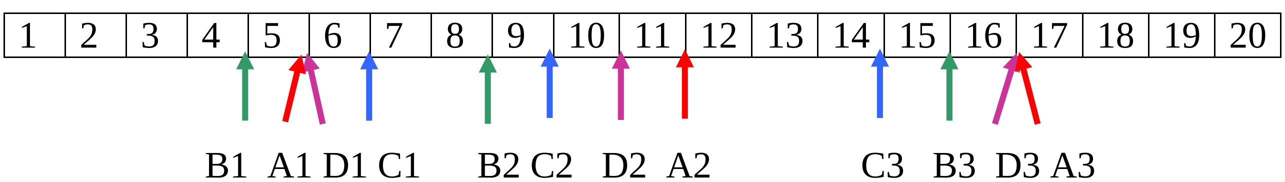


C's Markers



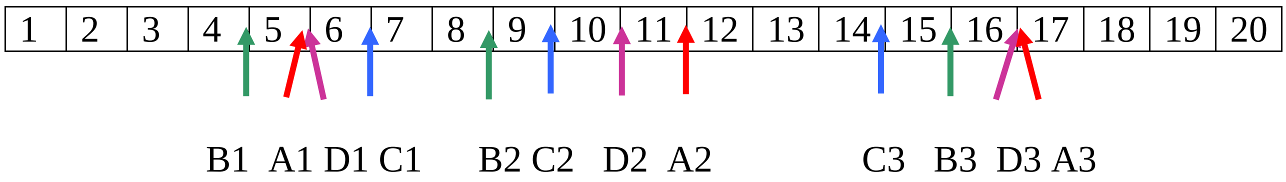
D's Markers

Summary of all 4 players' bids



(continues)

Summary of all 4 players' bids

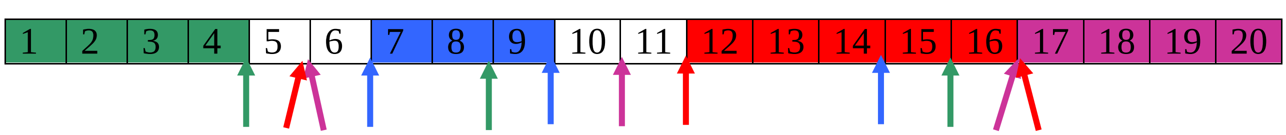


Step 2: Allocation: We look at the entire string, with each players' markers where they put them. Going left to right, we find the first marker. In this case the first marker is Bianca's (B1). Bianca gets the string up to this point. This includes **candies 1-4** inclusive. See diagram below. Then we remove or ignore all the rest of Bianca's markers.

Now scan left to right for the **first occurring second marker.** In this case, the first occurring second marker is Carla's C2. Carla gets the string **between C1 to C2**, or **candies 7-9**. Notice that after locating the first occurring second marker, we **assign the items going back to the left, to the corresponding first marker.**

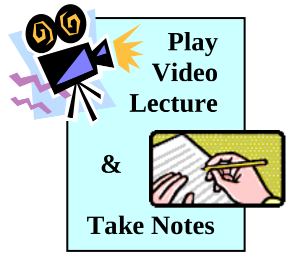
Then the first occurring of the remaining third markers is a tie, because both Alice and Dana have placed their markers in the same spot (A3 & D3). The tie is decided by tossing a coin. Alice wins the toss and the 3rd segment from A2 to A3, or **candies 12-16**. Again, notice that after locating the first occurring third marker, we **assign the items going back to the left, to the corresponding second marker.**

Finally, Dana receives the candy from marker D3 to the end, or **candies 17-20**. Notice that the last to receive a share receives those from that players last marker and on **to the right** to the end of the string of items.



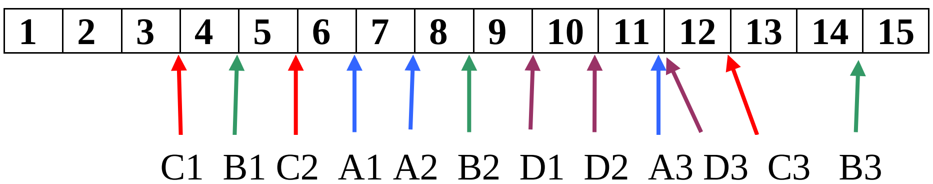
Notice that each colored region above corresponds to one of the strings of items that player bid on originally.

Step 3: Dividing the leftovers: Usually there are just a few pieces left, and they are distributed by drawing lots and letting the players choose 1 of the leftovers in the order of the lots.



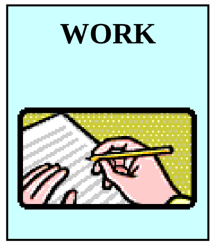
Problem 64 from Page 122 in Text

Four players: A, B, C, and D, agree to divide the 15 items shown below by lining them up in order and using the method of markers. The player's bids are as indicated.

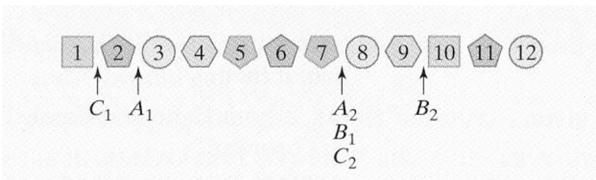


(a) Describe the allocation of items to each player.

(b) Which items are left over?



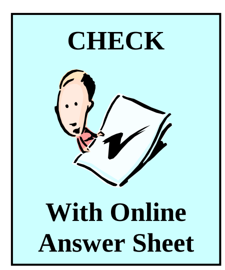
Problem 62 page 122. Three players (A, B, and C) agree to divide the 12 items shown by lining them up in order and using the method of markers. The players' bids are as indicated.



(a) Describe the allocation of items to each player.

(b) Which items are left over?

(c) Notice there is another way the items could be divided so that each player gets a fair share, but the division does not follow the method of markers. What is it?



End of Chapter Checklist

Before beginning the next Part be sure you have done the following:

_____1. Worked through all the guided notes for
Part 10: Fair Division

_____2. Worked through all the accompanying online video sessions for
Part 10: Fair Division

_____ 3. Checked all the "WORK" pages with the online answer sheets for
Part 10: Fair Division

_____4. Completed, checked and corrected all the homework problems from Chapter 3 the text **Pages 111-122**

_____5. Completed and turned in the **online quiz on Chapter 3** for
Part 10: Fair Division

_____ 6. Reviewed and studied for third unit exam

_____7. Completed Exam 3 over Parts 8 -10: Chapters 1 – 3

Part 11: Euler Circuits

AKA: Graph Theory or Routing Problems

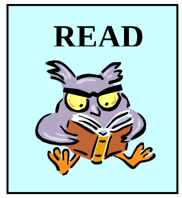
(Chapter 5 in textbook)

To the student:

- **This Part corresponds to Chapters 5 in the textbook**
- **First, read Chapter 5. Read for the BIG IDEAS and basic meaning. Don't worry about the details. We will do that in the guided notes.**
- **Then work through these guided notes**
- **Then do the homework assignment below**

  Homework: Pages 185-191 #1, 4, 9- 11, 17, 41- 47, 51

Associated Online Resources for Part 11 Guided Notes (GN)		
GN Page	**Type of Online Resource**	**Name of Online Resource**
263-265	**Video Lecture**	Vertices, Edges, Graphs
266	Answer Sheet	Practice with Vertices, Edges, and Graphs
268	Answer Sheet	Graph Terminology Practice
270-273	**Video Lecture**	Examples for Euler's Theorems
274	Answer Sheet	More Graph Terminology Practice
276	**Video Lecture**	Another Example of Representing a Map
277-278	**Video Lecture**	Fleury's Algorithm
279	**Video Lecture**	Eulerizing Graphs
280	Answer Sheet	Practicing Eulerizing Graphs
281	**Video Lecture**	Semi-Eulerizations
		Online Quiz Chapter 5



Section 5.1 Routing Problems

Routing Problem: finding a way to route the delivery of goods and/or services to an assortment of destinations.

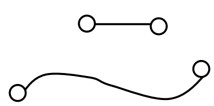
We will ask two types of questions:

(1) Is the delivery possible: Does a proper route exist?
(2) If there is more than one possible route, which one is the best? (Best Route is a function of some predetermined variable such as cost, distance, or time).

In real life, these would be the types of routing problems solved to create routes for garbage pickup, mail delivery, police patrol, meter readers, etc.

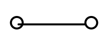
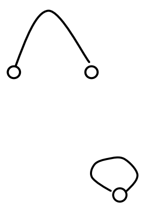
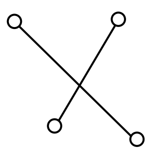
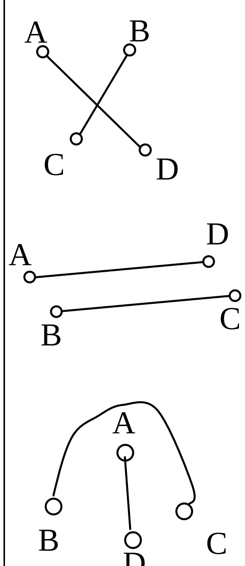
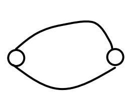
Read pages 179-183 of your text. These are a good introduction to the idea of Routing Problems.

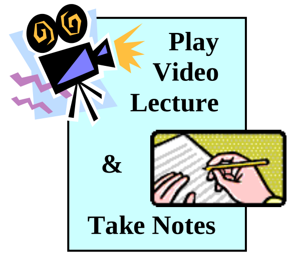
Section 5.2 Graphs

Picture	**Graph Technical Term**
(Dots) 	Vertices
(Connecting lines or curves) 	Edges
(Curve that connects back to the same dot) 	Loop

READ

Important Differences Between Graphs & High School Geometry

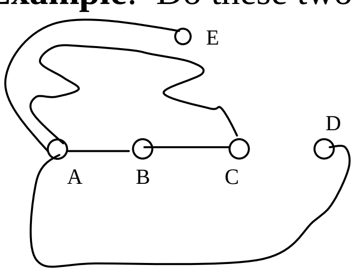
	Vertices exist ONLY where there are dots.
not a graph	There is no such thing as an edge with no vertices at its beginning and end.
	Edges can be curved and can ONLY exist between one vertex and another vertex OR from one vertex to itself.
	Just because 2 edges cross does NOT mean there is a vertex at what in high school we called the “intersection”. There are only 4 vertices in this graph.
	Shape is NOT the defining feature of a graph. It is the relationships of the edges connecting the vertices that defines the graph. These 3 pictures show are all “the same graph” because they show exactly the same relationship among vertices and edges: Each of these 3 graphs have the exact same makeup: 4 vertices and two edges with vertex A is only connected to vertex D and vertex B is only connected to vertex C. This makes them all equivalent representations of the “same graph”
	There can be more than one edge connecting the same two vertices.

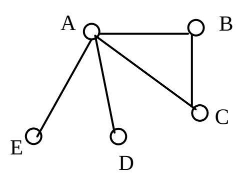


Vertices, Edges, and Graphs

Example: Draw 2 different pictures for the graph with vertices L, M, N, P and edges LP, MM, PN, MN, and PM.

Example: Do these two pictures represent the same graph or of different graphs?





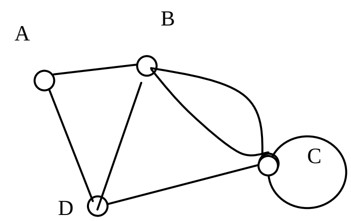
(continues)

Degree of a vertex: the number of edges that connect to that vertex.

Even Vertex: the degree of the vertex is an even number

Odd Vertex: the degree of the vertex is an odd number.

Example: For this graph, answer the following questions.



(a) How many vertices are there?

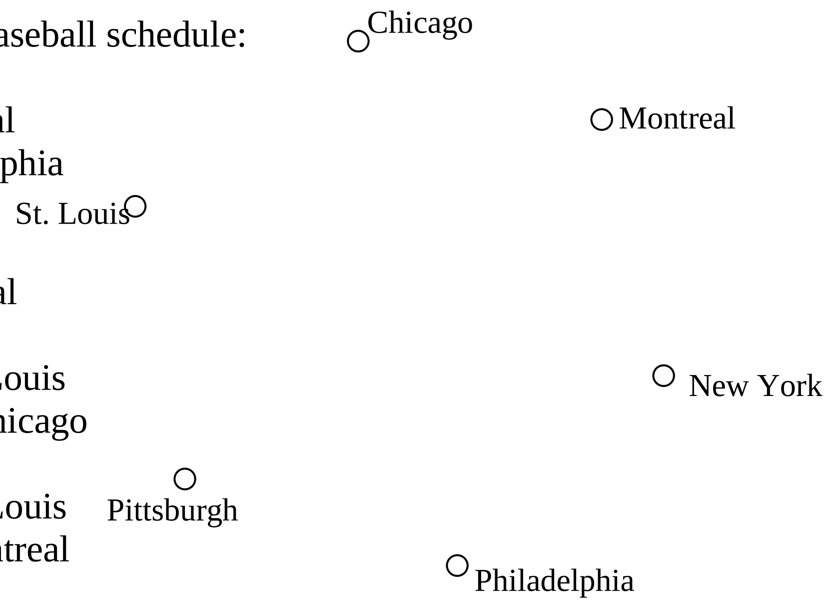
(b) How many edges are there?

(c) Find the degree of each vertex and whether it is odd or even.

(continues)

Example 5.10 A baseball schedule can be represented in terms of a graph showing which teams will play which other teams. In this case the vertices are the teams and the edges represent "will play against".

Make a graph of the following baseball schedule:



Monday: Pittsburgh vs. Montreal
New York vs. Philadelphia
Chicago vs. St. Louis

Tuesday: Pittsburgh vs. Montreal

Wednesday: New York vs. St. Louis
Philadelphia vs. Chicago

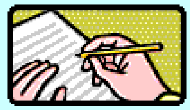
Thursday: Pittsburgh vs. St. Louis
New York vs. Montreal

Friday: Philadelphia vs. Montreal

Saturday: Philadelphia vs. Pittsburgh
New York vs. Chicago
Montreal vs. St. Louis

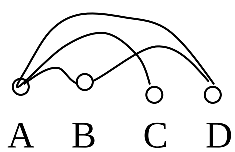
Sunday Philadelphia vs. Pittsburgh

WORK



Practice with Vertices, Edges, and Graphs

Problem 5a page 186 text



(a) List the edges in this graph.

(b) List the vertices in this graph and give the degree of each one.

Problem 4a page 186 text

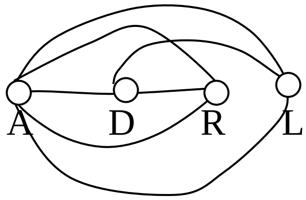
Draw two different pictures of the graph described:

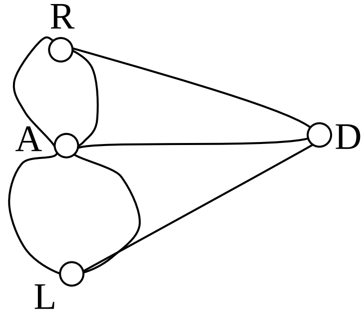
Vertices L, M, N, P

Edges LP, MM, PN, MN, PM

Example 5.9 page 167 text

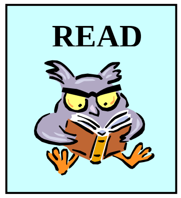
Determine whether these two graphs are representations of the same graph or not. Explain how you can tell.





CHECK

With Online Answer Sheet



Section 5.3 Graph Concepts and Terminology

Adjacent vertices: vertices that have an edge connecting them.

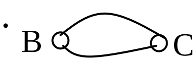
Adjacent edges: edges that share a common vertex

Degree of vertex: the number of edges that are connected to the vertex.

Odd vertex: a vertex with an odd number of connecting edges (degree is odd).

Even vertex: a vertex with an even number of connecting edges (degree is even).

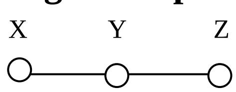
Multiple Edges: there can be more than one edge connecting the same two vertices. Example: there are two edges that connect B directly to C.



Path: sequence of vertices with property that each vertex in the sequence is adjacent to the next vertex in the sequence. A path can also be thought of as a sequence of adjacent edges.

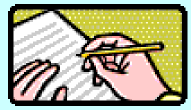
In a path <u>vertices</u> may be used <u>more than once</u>
BUT <u>edges</u> may be used <u>only once</u> in a path.

Length of a path: the number of edges in the path.



X, Y, Z is a path of length **2.**

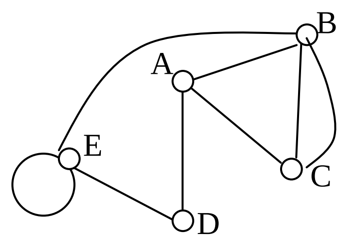
WORK



Graph Terminology Practice

Circle the choice in parentheses that correctly completes each statement.

Adjacent vertices: vertices that have an edge connecting them.



A (is adjacent/not adjacent) to B.

A (is adjacent/not adjacent) to D

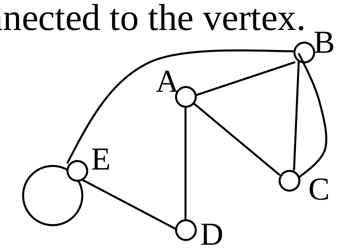
E (is adjacent/not adjacent) to C

Adjacent edges: edges that share a common vertex

BE (is adjacent/not adjacent) DE

AD (is adjacent/not adjacent)BE

Degree of vertex: the number of edges that are connected to the vertex.



Deg (A) =

Deg (B) =

Deg (E) =

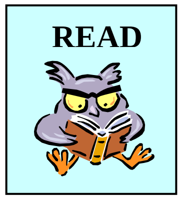
Odd vertex: a vertex with an odd number of connecting edges (degree is odd).
The odd vertices in the diagram above are:______________________

Path CABEEDA is a path of length ___________.

Is CABEEDABC a path? Why or why not?

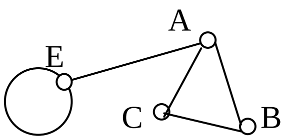
CHECK

With Online Answer Sheet

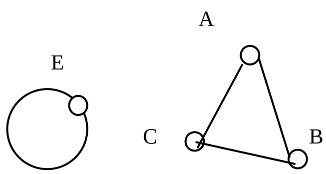


More Graph Terminology

Circuit: a path that starts and ends at the same vertex.



Connected graph: A graph is connected if any 2 of its vertices can be joined by a path. That is, no vertex or set of vertices is isolated from the rest of the graph.



Disconnected graph: A graph that is not connected.

Components of a graph: Disconnected graphs have more than 1 component. The graph above has two components. Each component is a piece of a graph that is connected (within itself). In contrast, a connected graph has only 1 component.

Bridge: an edge of a graph, that, if it we erased it, would disconnect the graph.

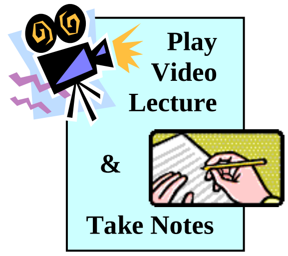
Euler Path: travels through EVERY edge of the graph ONCE and ONLY ONCE.

- The length of an Euler path = the number of edges in the graph.
- Same as a unicursal tracing – that is, a graph that can be traced completely without lifting the pencil and without retracing any of the lines in the process.
- Not every graph has an Euler path.

Euler Circuit: an Euler path that begins and ends at the same vertex.

- Same as a closed unicursal tracing

(continues)



Examples for Euler's Theorems

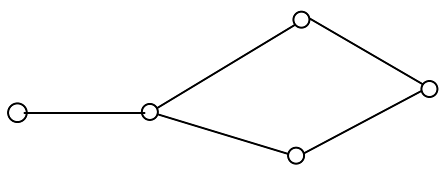
Euler Path: travels through EVERY edge of the graph ONCE and ONLY ONCE.

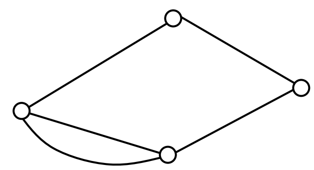
Euler Circuit: an Euler path that begins and ends at the same vertex.

- Same as a closed unicursal tracing

Euler's Theorem 1

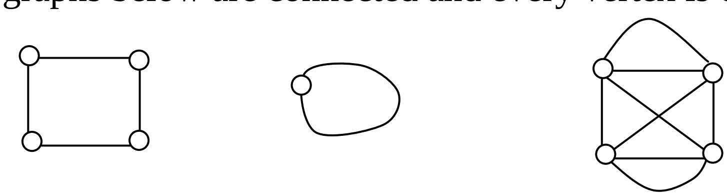
(a) if a graph has ***any odd vertices***, then it cannot have an Euler circuit



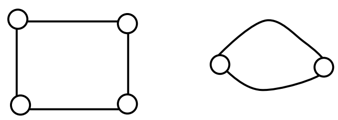


(b) if a graph is ***connected*** and ***every vertex is an even vertex***, then it has at least one Euler circuit (and usually more).

(i) 3 graphs below are connected and every vertex is even



(ii) 1 graph below has every vertex even but is not connected



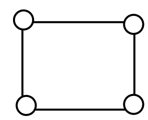
(continued)

Euler Path: travels through EVERY edge of the graph ONCE and ONLY ONCE.

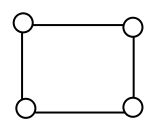
Euler's Theorem 2

(a) If a graph has ***more than two odd vertices***, then it cannot have an Euler path

(i) try drawing a graph that has exactly 1odd vertice.

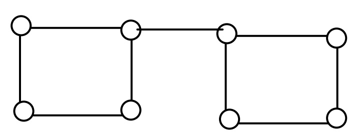


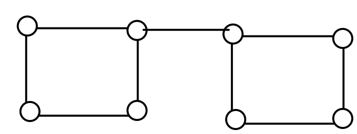
(ii) more than 2 odd vertices

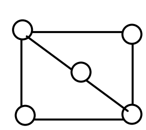


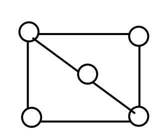
(b) If a graph is connected and has ***exactly two odd vertices***, then it has at least one Euler path (and usually more). Any such path must start at one of the odd vertices and end at the other one.

(i) can you find an Euler path? (ii) begin and end with the odd vertices









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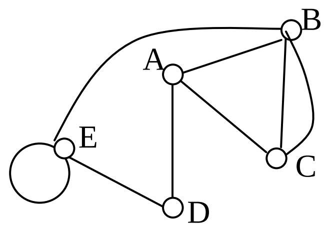
Euler's Theorem 3

(a) The sum of the degrees of all the vertices of a graph equals twice the number of edges (and therefore is an even number).



(b) A graph always has an even number of ***odd*** vertices.

Summary of Euler Theorems

Number of Odd Vertices	**Conclusion**
0	Graph has one or more Euler Circuits
2	Graph has one or more Euler Paths but no Euler Circuits
4, 6, 8, . . .	Graph has neither Euler Paths nor Euler Circuits
1, 3, 5, . . .	Impossible. No such graphs exist

(continues)

Unicursal Tracings

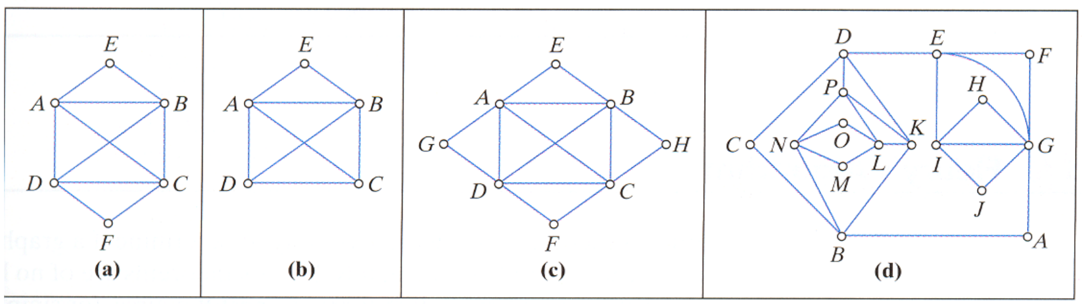
Closed unicursal tracing: The figure may be traced without lifting the pencil or retracing a line, and end at the same point where you began.

This is the same as a ________________________________.

Open unicursal tracing: The figure may be traced without lifting the pencil or retracing a line. It is not necessary to begin and end at the same point.

This is the same as a _________________________________.

Figure 5-16 page 173 of your textbook

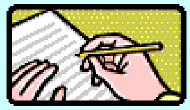


Which of the above graphs have closed unicursal tracings?___________________

Which of the following have open unicursal tracings? ______________________

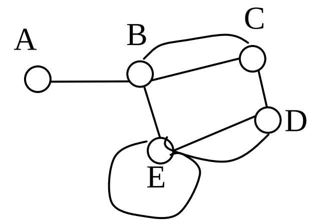
Which cannot be traced either way? _____________________________________

WORK



More Graph Terminology Practice

1. List all the edges that are adjacent to edge BC in the graph at the right.



2. List all the edges that are multiple edges in the graph above.

3. Name a vertex that has a loop._______ what is the degree of this vertex?_____

4. Name the bridge in the graph above.

5. How many components does this graph have?

6. How long is the path BCDEBA?

6. Is it possible to find an Euler circuit for this graph? Why or why not?

CHECK

With Online Answer Sheet

READ

Seven Bridges of Konigsberg Problem

(pages 170-171, 173 in textbook)

A true story: Seven gold coins were offered as a prize for the first person who could find a way to walk across each one of the seven bridges of Konigsberg without recrossing any and return to the original starting point. A smaller prize (5 gold coins) was offered for anyone who can cross each of the seven bridges exactly once without necessarily returning to the starting point. No one has ever collected either prize.

Figure 5-12 of your textbook, (reproduced below) shows a map of the Konigsberg bridge area. Notice that the river splits and there are two islands in the middle of the split. The seven bridges are shown in part (a) below by the dark black lines.

Part (b) and (c) of figure 5-12 shows the situation simplified as a “graph” representation of the land masses (represented by vertices) and the bridges (represented by edges).

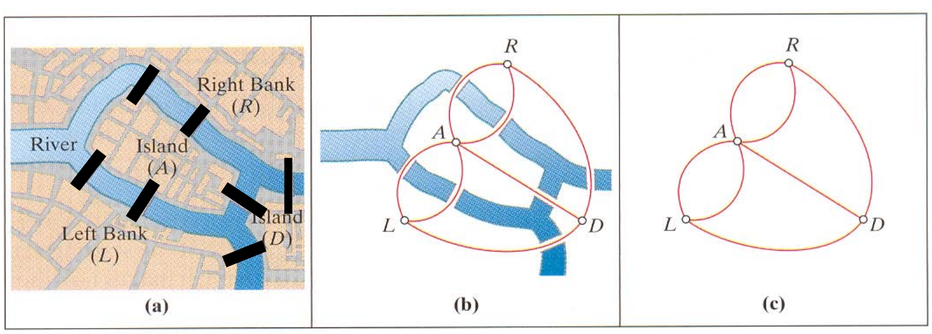


FIGURE 5-14

Notice that all 4 of the vertices in part (c) above are vertices with odd degree. When the mathematician Euler proved that a graph with more than 2 odd vertices would not have an Euler circuit, he proved that it was not possible to solve the Konigsberg Bridge challenge. No one will ever be able to win the prize.

This is an example of how real path or schedule problems can be solved (or proved impossible to solve) using the mathematics of graph theory.

Another Example of Representing a Map as a Graph

Examples 5.1, 5.2, 5.12, and 5.13 in your textbook (pp. 162-163 and p. 171 show another routing situation that can be represented and solved as a Graph.

A security patrol going down the middle of the street effectively patrols both sides of the street at once. The mail carrier, however, must deliver to both sides of the street whenever there are houses on both sides of the street.

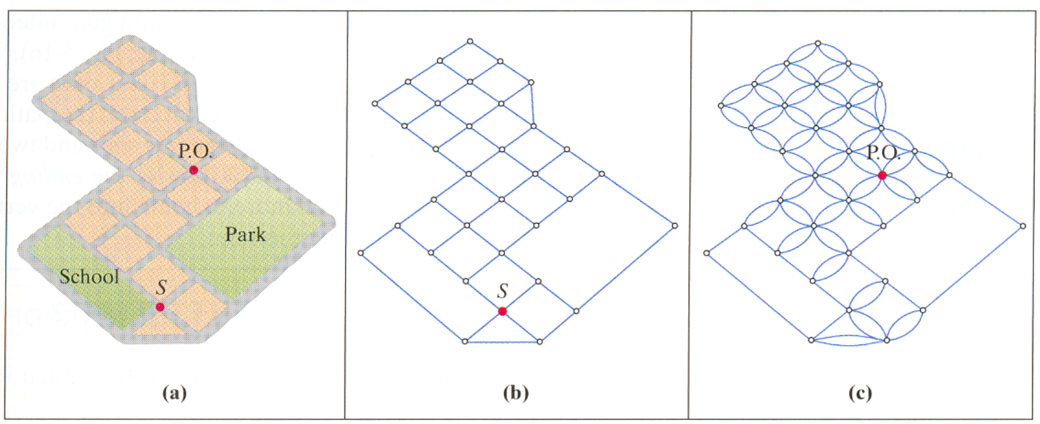


FIGURE 5-15

The vertices represent ________________________________.

The edges represent __________________________________.

The difference between graph (b) and graph (c) represents ___________________

____

Fleury's Algorithm for Finding an Euler Circuit

Steps for Fleury's Algorithm:

Step 1: First make sure that the graph is connected and all the vertices have even degree.

Step 2: Pick any vertex as the starting point.

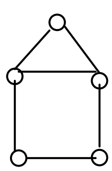
Step 3. When you have a choice, always choose to travel along an edge that is not a bridge of the yet-to-be-traveled part of the graph. (Don't burn your bridges!!!!)

Step 4. Label the edges in the order in which you travel them.

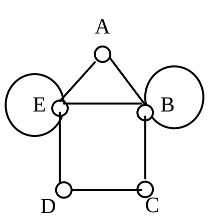
Step 5. When you can't travel any more, stop. You are done!

Example: Apply the Fleury Algorithm to each of the figures below. Write out any Euler Circuit that you find.

(a)



(b)



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Example Use Fleury's algorithm to find an Euler Circuit for this graph.

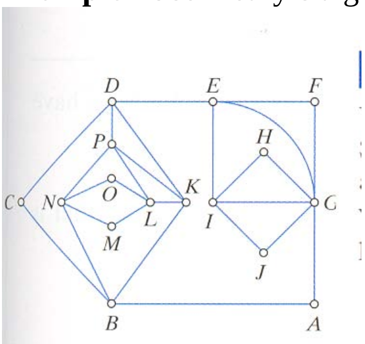
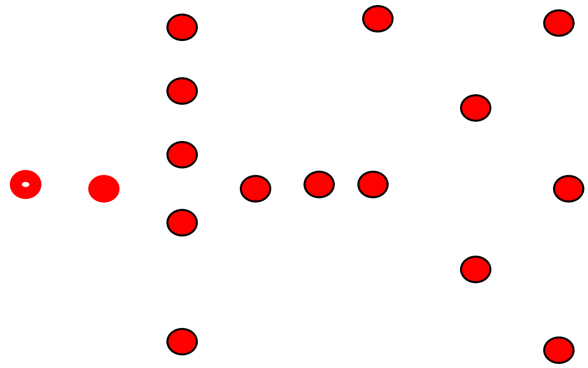
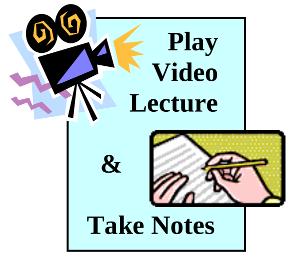


FIGURE 5-21





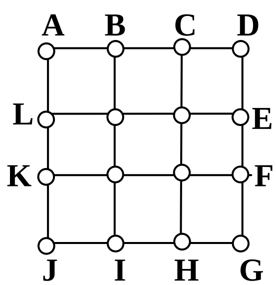
Eulerizing Graphs (section 5.7 in text)

Since graphs with more than 2 odd vertices will have no Euler Circuit or Path, finding an optimal route in these cases requires that we recross some of the edges. The question becomes how can we cover all the edges of the graph with only recrossing the minimum number of edges. This process is called "Eulerizing" a Graph. When Eulerizing a graph, we add edges that meet two requirements:

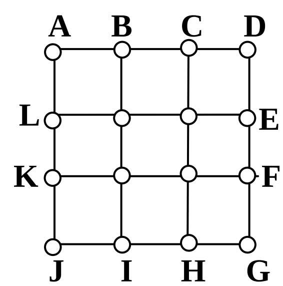
(a) only add edges that duplicate already existing edges
(b) only add edges beginning and ending at odd vertices.

Example 5.19 in text:

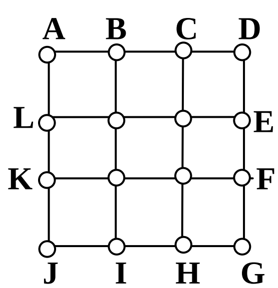
Step 1:
Original Graph



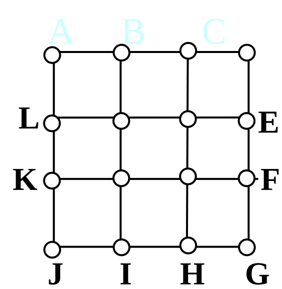
Step 2:
Eulerized Graph



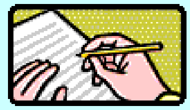
Step 3:
Euler Circuit on Eulerized Graph



Step 4:
Euler Circuit applied to Original Graph

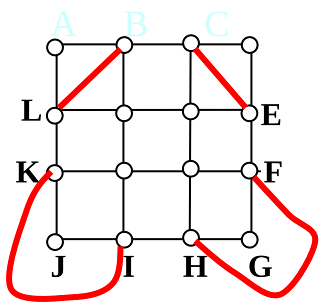


WORK

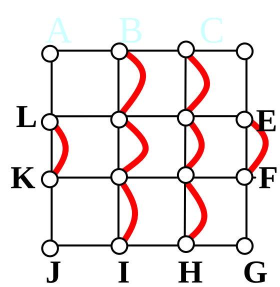


Practice Eulerizing Graphs

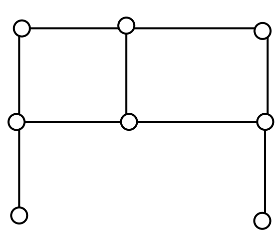
1. The following is not a legal Eulerization of the original graph.
   What is wrong with it?



2. The following is not an efficient Eulerization of the original graph.
   What makes it inefficient?



3. Make a legal, efficient Eulerization of this graph:



4. Write out an Euler Circuit for the Eulerized Graph above.

CHECK

With Online Answer Sheet

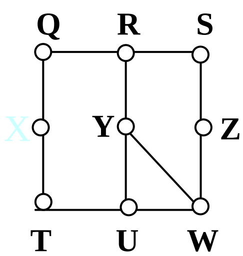
Semi-Eulerization of Graphs

A Semi-Eulerization of a graph makes all but two of the vertices even, so that the graph can be modified to make a Euler Path (as opposed to Euler Circuit)

The difference is that the path does not have to start and end at the same vertex.

The same rules apply to creating Semi-Eulerizations as to creating Eulerizations.

Remember that the Euler Path for such a graph must start at one of the odd vertices and end at the other odd vertex.



End of Chapter Checklist

Before beginning the next Part be sure you have done the following:

_____1. Worked through all the guided notes for
Part 11: Euler Circuits

_____2. Worked through all the accompanying online video sessions for
Part 11: Euler Circuits

_____ 3. Checked all the "WORK" pages with the online answer sheets for
Part 11: Euler Circuits

_____4. Completed, checked and corrected all the homework problems from
Chapter 5 the text **Pages 185 – 191 #1, 4, 9- 11, 17, 41- 47, 51**

_____5. Completed and turned in the **<u>online quiz on Chapter 5</u>** for
Part 11: Euler Circuits

.

Part 12: The Traveling-Salesman Problem

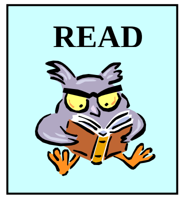
(Chapter 6 in textbook)

To the student:

- **This Part corresponds to Chapters 6 in the textbook**
- **First, read Chapter 6. Read for the BIG IDEAS and basic meaning. Don't worry about the details. We will do that in the guided notes.**
- **Then work through these guided notes**
- **Then do the homework assignment below**

  Homework: Pages 221 – 226 #1, 4, 9, 10, 14, 29, 33, 35, 37, 38, 43, 45, 49, 51

Associated Online Resources for Part 12 Guided Notes (GN)		
GN Page	**Type of Online Resource**	**Name of Online Resource**
285	**Video Lecture**	Hamilton and Complete Circuits
286	Answer Sheet	Problem 20 page 251
289-292	**Video Lecture**	A Tale of Five Cities
293-294	Answer Sheet	Problem 12 page 250
		Online Quiz Chapter 6
Final Exam over Parts 1-12		



Traveling Salesman Problems (TSP)

In these routing problems, we consider both distance and cost. It may be that different legs of the route have the same length, but not the same cost. The "traveling salesman" wants the route that allows him to visit each of the locations at the least cost for the total trip.

For these types of problems we will be looking for Hamilton Circuits or Paths.

Hamilton Circuit: a circuit that visits each **vertex** exactly once (except the first vertex which is also where the circuit ends). The difference between an Euler circuit and a Hamilton circuit is that Euler circuits cross each edge exactly once while a Hamilton circuits visit each vertex exactly once.

Hamilton Path: visits each vertex exactly once but does not return to the first vertex like a circuit does.

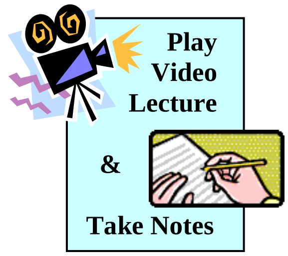
There is no fixed relationship between whether a graph has Hamilton and/or Euler circuits. A graph may have

- a Hamilton circuit and an Euler circuit
- a Hamilton circuit but not an Euler circuit
- an Euler circuit but not a Hamilton circuit
- Neither a Hamilton nor an Euler circuit

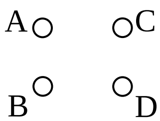
Complete Graph: A graph with N vertices in which every pair of vertices is joined by exactly one edge is called a complete graph.

$\mathbf{K_N}$ is the notation for the complete graph with N vertices.

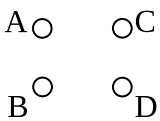
- Each vertex will have degree N-1.
- The total number of Edges will be $\frac{N(N-1)}{2}$
- A complete graph has (N-1)! Hamiltonian Circuits .



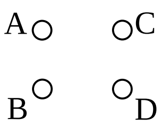
Hamilton and Complete Circuits



(1) Draw a Hamiltonian Path from A to D.



(2) Draw a Hamiltonian Circuit for these four vertices.



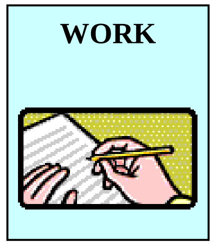
(c) Make the complete graph K_4 for these four vertices.

(d) How many edges will K_4 have?

(e) How many distinct Hamiltonian Circuits will K_4 have?

(f) Use the K_4 graph to write out all the Hamiltonian Circuits.
Use the form that begins the circuit with vertex A.

(g) If K_N has 120 distinct Hamiltonian circuits. Find N.

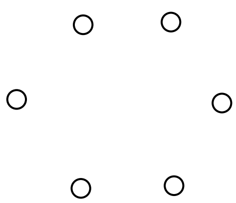


1. For the complete graph with 24 vertices, find

(a) the number of edges in the graph

(b) the number of distinct Hamilton circuits

2. (a) Draw all the edges to make the following a complete graph:

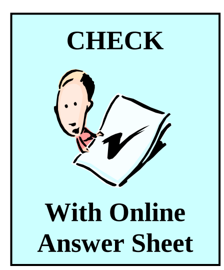


(b) The graph above has:

_______________ vertices

_______________ edges

________________ Hamiltonian Circuits



Algorithms for Finding Optimal and Approximately Optimal Hamiltonian Circuits

Brute Force Algorithm:

1. Make a list of all the possible Hamilton circuits of the graph.
2. Compute the total weight for each Hamilton circuit.
3. Find the circuits with the least weight.

characteristics of Brute Force Algorithm:

- inefficient, you must try every possible Hamilton circuits – this is only possible for graphs with few vertices
- guaranteed to find optimal circuit

Nearest-Neighbor Algorithm:

1. Pick a vertex as the starting point.
2. Go from this vertex to its nearest neighbor – the vertex for which the corresponding edge has the smallest weight.
3. Continue on, each time selecting the edge with the smallest remaining weight.
4. When all vertices have been used, take the edge that returns to the starting vertex.

characteristics of Nearest Neighbor Algorithm:

- efficient – uses only a few steps
- does not guarantee that the circuit you find is the one of very least weight – does not guarantee finding an optimal circuit.
- this is an approximate algorithm: it produces a solutions that are, most of the time, reasonably close to the optimal solution.

(continues)

The Repetitive Nearest-Neighbor Algorithm:

1. Start with one vertex, call it X, and apply the nearest-neighbor algorithm using X as the starting vertex and calculate the total cost of the circuit obtained.
2. Repeat the process using each of the other vertices of the graph as a starting vertex.
3. Of the Hamilton circuits obtained, keep the best one. If there is a designated starting point, rewrite the circuit with the vertex as the reference point.

Characteristics of the Repetitive Nearest-Neighbor Algorithm

- Looks at more possibilities and therefore is more likely to find a better choice than the simple Nearest-Neighbor Algorithm. (Is often a better approximation)
- Still is not guaranteed to find the optimal circuit.

The Cheapest-Link Algorithm:

1. Pick the edge with the smallest weight first. Mark it (for instance in red).
2. Pick the next "cheapest" edge and mark the edge in red.
3. Continue picking the "cheapest" edge available and mark the edge in red except when
   (a) it closes a circuit
   (b) it results in three edges coming out of a single vertex
4. When there are no more vertices to join, close the red circuit.

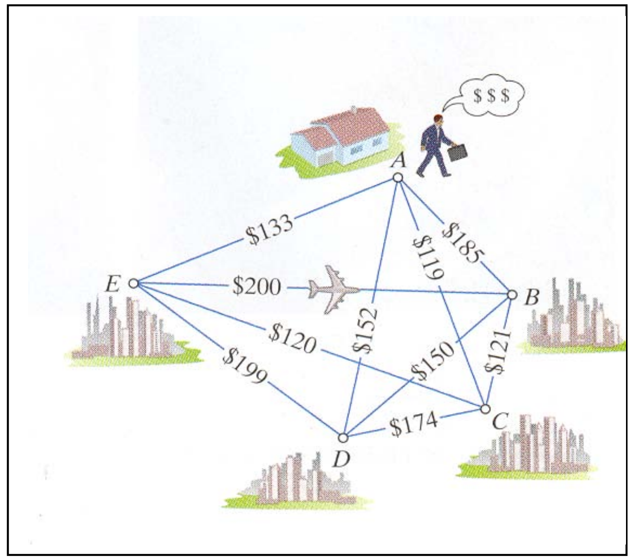
Characteristics of the Cheapest-Link Algorithm:

- It is still an approximation and not guaranteed to find the optimal circuit.
- It has a limited number of steps (efficient).

The Mathematical Point of the Story: There has not yet been found an efficient algorithm that guarantees finding the optimal circuit in a TSP (Traveling Salesman-type Problem). You have to make a choice between inefficiency (Brute Force Algorithm) and approximation (all the other algorithms).

A Tale of Five Cities

1. Brute Force Method



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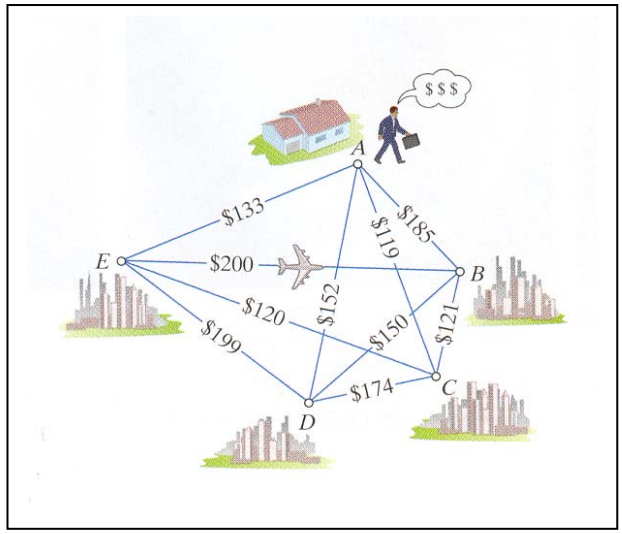
2. Nearest-Neighbor Algorithm



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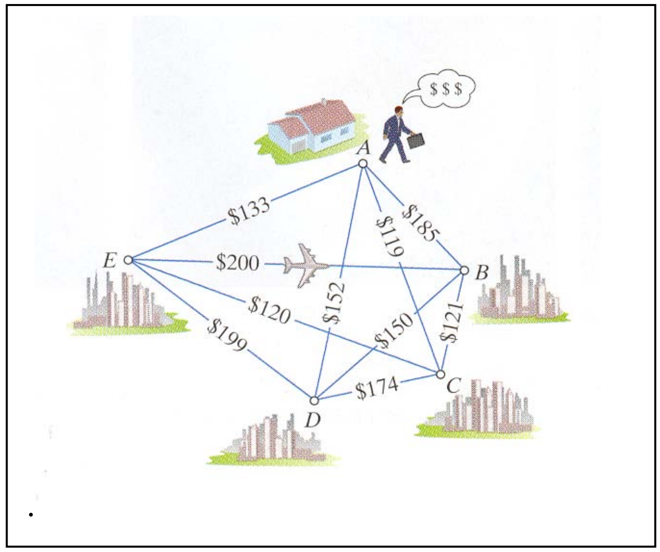
3. The Repetitive Nearest-Neighbor Algorithm



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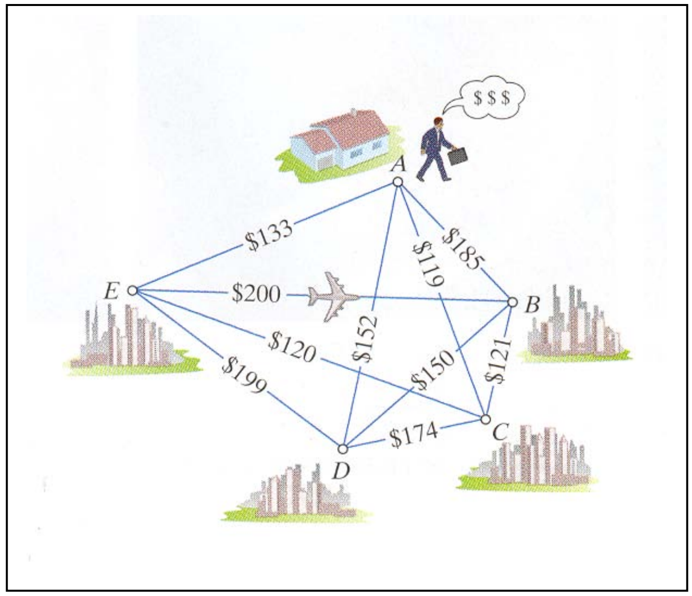
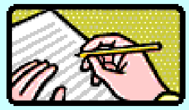
4. Cheapest Link Algorithm



WORK



Copy out the diagram for problem 14 on page 222 of your textbook here:

Problem 14 page 222 (modified) : For the weighted graph above

(a) Find the weight of edge AD

(b) Find the weight of edge AC

(c) find the weight of the Hamilton Circuit ABCDEA.

(d) find the weight of the Hamilton Circuit ACBEDA

(e) find the weight of the Hamilton Circuit AECBDA.

(continues)

Problem 30 page 224. Copy the diagram for this problem out of your text and then answer parts (a) and (b).

CHECK

With Online Answer Sheet

End of Chapter Checklist

This is the last Part of the Course MA 105 Contemporary Mathematics. Be sure you have:

_____1. Worked through all the guided notes for
Part 12: The Traveling Salesman Problem

_____2. Worked through all the accompanying online video sessions for
Part 12: The Traveling Salesman Problem

_____ 3. Checked all the "WORK" pages with the online answer sheets for
Part 12: The Traveling Salesman Problem

_____4. Completed, checked and corrected all the homework problems from Chapter 6 the text
Homework: Pages 221 – 226 #1, 4, 9, 10, 14, 29, 33, 35, 37, 38, 43, 45, 49, 51

_____5. Completed and turned in the **online quiz on Chapter 6** for
Part 12: The Traveling Salesman Problem

_____6. Final EXAM over Parts 1-12