1.1.1 Introduction to Axiomatic Systems

Words differently arranged have a different meaning and meanings differently arranged have a different effect.
—Blaise Pascal (1623–1662)

Axiomatic System (Postulate System)

1. Undefined terms/primitive terms
2. Defined terms
3. Axioms/postulates - accepted unproved statements
4. Theorems - proved statements

An *axiomatic system* consists of some undefined terms (primitive terms) and a list of statements, called *axioms* or *postulates*, concerning the undefined terms. One obtains a mathematical theory by proving new statements, called *theorems*, using only the axioms (postulates), logic system, and previous theorems. Definitions are made in the process in order to be more concise.

Most early Greeks made a distinction between axioms and postulates. Evidence exists that Euclid made the distinction that an axiom (common notion) is an assumption common to all sciences and that a postulate is an assumption peculiar to the particular science being studied. Now in modern times no distinction is made between the two; an axiom or postulate is an assumed statement.

Usually an axiomatic system does not stand alone, but other systems are also assumed to hold. For example, we will assume:

1. the real number system,
2. some set theory,
3. Aristotelian logic system, and
4. the English language.

We will not develop any of these but use what we need from them.

One of the pitfalls of working with a deductive system is too great a familiarity with the subject matter of the system. We need to be careful with what we are assuming to be true and with saying something is obvious while writing a proof. We need to take extreme care that we do not make an additional assumption outside the system being studied. A common error in the writing of proofs in geometry is to base the proof on a picture. A picture may be misleading, either by not covering all possibilities, or by reflecting our unconscious bias as to what is correct. *It is crucially important in a proof to use only the axioms and the theorems which have been derived from them and not depend on any preconceived idea or picture.* Pictures should only be used as an intuitive aid in developing the proof, but each step in the proof should depend only on the axioms and the theorems with no dependence upon any picture. *Diagrams should be used as an aid, since they are useful in developing conceptual understanding, but care must be taken that the diagrams do not lead to misunderstanding.* Two exercises in Chapter Two illustrate this point: (1) A false proof that all triangles are isosceles. (2) A faulty proof of a valid theorem.

Usually not all the axioms are given at the beginning of the development of an axiomatic system; this allows us to prove very general theorems which hold for many axiomatic systems. An example from abstract algebra is: group theory → ring theory → field theory. A second example is a parallel postulate is often not introduced early in studies of Euclidean geometry, so the theorems developed will hold for both Euclidean and hyperbolic geometry (called a neutral geometry).

Certain terms are left undefined to prevent circular definitions, and the axioms are stated to give properties to the undefined terms. Undefined terms are of two types: terms that imply objects, called *elements*, and terms that imply relationships between objects, called *relations*. Examples of undefined terms (primitive terms) in geometry are point, line, plane, on, and between. For these undefined terms, on and between would indicate some undefined relationship between undefined objects such as point and line. An example would be: A point is on a line. Early geometers tried to define these terms:

point Pythagoreans, “a monad having position"
Plato, “the beginning of a line"
Euclid, “that which has no part"

line Proclus, “magnitude in one dimension", “flux of a point"
Euclid, “breadthless length"

Euclid made the attempt to define all of his terms. (See *Euclid's Elements*.) Now, points are considered to come before lines, but no effort is made to define them a priori. Instead, material things are used as illustrations/models to obtain the abstract idea. The famous mathematician David Hilbert (1862–1943) is quoted as saying, “we may as well be talking about chairs, coffee tables and beer mugs."

An axiomatic system is *consistent* if there is no statement such that both the statement and its negation are axioms or theorems of the axiomatic system. Since contradictory axioms or theorems are usually not desired in an axiomatic system, we will consider consistency to be a necessary condition for an axiomatic system. An axiomatic system that does not have the property of consistency has no mathematical value and is generally not of interest.

A *model* of an axiomatic system is obtained if we can assign meaning to the undefined terms of the axiomatic system which convert the axioms into true statements about the assigned concepts. Two types of models are used *concrete models* and *abstract models*. A model is concrete if the meanings assigned to the undefined terms are objects and relations adapted from the real world. A model is abstract if the meanings assigned to the undefined terms are objects and relations adapted from another axiomatic development.

Consistency is often difficult to prove. One method for showing that an axiomatic system is consistent is to use a model. When a concrete model has been exhibited, we say we have established the *absolute consistency* of the axiomatic system. Basically, we believe that contradictions in the real world are impossible. If we exhibit an abstract model where the axioms of the first system are theorems of the second system, then we say the first axiomatic system is *relatively consistent*. Relative consistency is usually the best we can hope for since concrete models are often difficult or impossible to set up. An axiomatic system is *complete* if every statement containing the undefined and defined terms of the system can be proved valid or invalid. Also, Kurt Gödel (1906–1978) with his Incompleteness Theorem (published in 1931 in *Monatshefte für Mathematik und Physik*) demonstrated that even in elementary parts of arithmetic there exist propositions which cannot be proved or disproved within the system.

In an axiomatic system, an axiom is *independent* if it is not a theorem that follows from the other axioms. Independence is not a necessary requirement for an axiomatic system; whereas, consistency is necessary. For example, in high school geometry courses, theorems which are long and difficult to prove are usually taken as axioms/postulates. Hence in most high school geometry courses, the axiom sets are usually not independent. In fact, in this course, though we will be much more rigorous than in a high school course, we may at times take some theorems as postulates.

Many people throughout history have thought that Euclid's Fifth Postulate (parallel postulate) was not independent of the other axioms. Many people tried to prove this axiom but either failed or used faulty reasoning. This problem eventually led to the development of other geometries, and Euclid's Fifth Postulate was shown to be independent of the other postulates. We will not be assuming the parallel postulate at the beginning of our study of Euclidean geometry; this will allow us to develop many theorems which are valid in some non-Euclidean geometries.

Models of an axiomatic system are *isomorphic* if there is a one-to-one correspondence between their elements that preserves all relations. That is, the models are abstractly the same; only the notation is different. An axiomatic system is *categorical* if every two models of the system are isomorphic.

In a geometry with two undefined primitive terms, the *dual* of an axiom or theorem is the statement with the two terms interchanged. For example, the dual of "A line contains at least two points," is "A point contains at least two lines." An axiom system in which the dual of any axiom or theorem is also an axiom or theorem is said to satisfy the *principle of duality.* Plane projective geometry, which we will study later in the course, is an example of a geometry that satisfies the principal of duality.

God exists since mathematics is consistent, and the devil exists since we cannot prove the consistency.
—Morris Kline (1908–1992)

1.1.2 Examples of Axiomatic Systems

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