1.1.3 Historical Overview Axiomatic System and Finite Geometry

Mathematicians do not study objects, but relations among objects; they are indifferent to the replacement of objects by others as long as the relations don't change. Matter is not important, only form interests them.
—Henri Poincaré (1854–1912)

As we look through a history of the study of axiomatic systems and finite geometries, it is important to have a slight understanding of what an axiomatic system is and how it began. An axiomatic system is a logic structure in which we prove statements from a set of assumptions. Axiomatic systems consist of four main parts: undefined terms, defined terms, axioms/postulates (accepted or unproven statements), and proved statements. Starting with undefined terms and a list of statements, called axioms and postulates, one is able to obtain new theorems by proving statements using only the axioms or postulates and previously proved theorems. Historically, when axiomatic systems were first being studied, distinctions were made between axioms and postulates. The word postulate was used by Euclid and other early Greek mathematicians to stand for an assumed truth peculiar to one particular science, while an axiom was used as an assumption common to all sciences. This difference was commonly made by early Greek mathematicians. An example of the way these terms were used is shown in Euclid's book the *Elements* where Euclid made distinctions between postulates and common notions (axioms). Today, however, both terms are used interchangeably to refer to an assumed statement.

Undefined terms are included in axiomatic systems from historical problems that early geometers ran into. Geometers, such as Proclus, Euclid, and Plato, tried to define most terms in their work. Common examples of these terms that they attempted to define were point, line, and plane; however, they quickly ran into problems. While attempting to define these terms, geometers would frequently run into the problem of needing to define a term with another term. This process would lead to circular definitions or an infinite chain of terms and their definitions. An example of Euclid's use of circular definitions can be seen in Euclid's definitions of point and line; Euclid defined a point as 'that which has no part' and a line as 'length without breadth.' The questions then arise: What is 'no part'? What is 'length'? What is 'breadth'? From this common fault that mathematicians ran into, the mathematician, David Hilbert, was quoted as saying, "we may as well be talking about chairs, coffee tables and beer mugs."

David Hilbert (1862–1943), a German number theorist and mathematician, is one of the best known mathematicians. Hilbert began his career in algebra and number theory, and then moved on to study geometry. His work in geometry was based on Euclid's work in geometry from about 2000 years earlier. Euclid's proofs, although revolutionary for his time, did contain gaps where he had made tacit assumptions, or assumptions that were not mathematically warranted. One such example is seen in "the proof of Proposition I 16, where Euclid unconsciously assumed the infinitude of straight lines" (Eves 457). Hilbert's early work with Euclid's proofs and axiomatic systems "has been more helpful in enabling mathematicians to pursue the foundations of geometry" (Young 246). His work transformed the flawed method used in Euclid's proofs to the format of axiomatic systems used today.

An exciting time for Hilbert came in 1900 when he was asked to speak at the International Congress of Mathematicians in Paris. During his speech, he proposed 23 problems he challenged fellow mathematicians to solve in the upcoming century. His professional life was remarkable; however, during the end of his life, his career was "disappear(ing) under the ideological onslaught from the Nazi government" during World War II (Young 245). This political government was the reason that this distinguished mathematician's funeral was attended by less than 12 people.

In 1904, Hilbert went to work on investigating mathematic logic and proving the consistency of mathematics. His ideas triggered tension between himself and other mathematicians, the most notable being

Kurt Gödel.

Kurt Gödel (1906–1978), an Austrian mathematician, is well-known for his Incompleteness Theorem, which exposed Hilbert's ideas. The Incompleteness Theorem was published in 1931 in *Monatsheft für Mathematik und Physik*, a German mathematical journal, with the title "On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems." This theorem demonstrated that even in elementary parts of arithmetic there exist propositions which cannot be proved or disproved within the system.

Gödel's mathematical career and life was also affected by the Nazi government, however, not to the same extent that Hilbert faced. Gödel decided to leave Europe and the threat of the Nazi government in 1939 after receiving a letter accusing him of interacting with Jews. After arriving in the United States, Gödel remained active with mathematics and became close friends with Albert Einstein.

Finite geometry followed the axiomatic systems in the late 1800's. Finite geometry was developed while attempting to prove the properties of consistency, independence, and completeness of an axiomatic system. Geometers wanted models that fulfilled specific axioms. Often the models found had finitely many points which contributed to the name of this branch of geometry.

Gino Fano (1871–1952), an Italian mathematician, is credited with being the first person to work with finite geometry (in 1892). His first work in this new geometry included 15 points, 35 lines, and 15 planes, all in a three-dimensional plane. Fano, like Hilbert and Gödel and many Europeans during the 1930's and 40's, was affected by the Nazi government. Fano also was forced to leave Europe but did continue his mathematics career and began to teach in the United States.

Even with Fano's early work, it wasn't until the early 1900's that finite geometry obtained a well-known role in mathematics. Considering the relatively short history of finite geometries, there are still unsolved problems actively being researched by leading mathematicians today.