2.1.1 Introduction to Euclidean and Non-Euclidean Geometry

Without the concepts, methods and results found and developed by previous generations right down to Greek antiquity one cannot understand either the aims or achievements of mathematics in the last 50 years.
—Hermann Weyl (1885–1955)

Euclid's Fifth Postulate, the parallel postulate, caused much dispute over many centuries. Many believed the postulate should be a theorem and not an assumption. Eventually, non-Euclidean geometries, based on postulates that were negations of Euclid's Fifth Postulate, were formulated.

Euclid the Thirteen Books of the Elements translated by Thomas Heath is the best English translation used today of Euclid's Elements. Here are links to two on-line editions of *Euclid's Elements*: David E. Joyce's Java edition of Euclid's Elements (1997) or Oliver Byrne's edition of Euclid published in 1847. The problem with using Euclid's five axioms as a basis for a course in Euclidean geometry is that Euclid's system has several flaws: Euclid tried to define all terms and did not recognize the need for undefined terms. Euclid made other assumptions based on preconceptions that were not stated as postulates. And, many proofs rely on diagrams and preconceptions about the diagrams.

Two different, but equivalent, axiomatic systems are used in the study of Euclidean geometry—synthetic geometry and metric geometry. David Hilbert (1862–1943), in his book *Gundlagen der Geometrie* (*Foundations of Geometry*), published in 1899 a list of axioms for Euclidean geometry, which are axioms for a synthetic geometry. Hilbert's axioms are in Appendix A of this chapter. George Birkhoff (1884–1944) in a paper (A set of postulates for plane geometry published in *Annals of Mathematics* in 1932) proposed a list of axioms for Euclidean geometry, which were axioms for a metric geometry. Birkhoff's axioms are in Appendix B of this chapter. Read through and compare their axioms for Euclidean geometry. How are they similar? How different?

The questions are: What axiom system should we use in this course? How should we study them? Since this course is a survey course for preparing teachers to teach high school geometry, we will use the SMSG axiom set, which was developed by the School Mathematics Study Group in 1961 as a suggestion for use with high school geometry courses. Note how the SMSG axioms are a blend of Hilbert's and Birkhoff's axioms. To show the similarities between Euclidean and non-Euclidean geometries, we will postpone the introduction of a parallel postulate to the end of this chapter. We will study what is called *neutral geometry*, the properties of which satisfy both Euclidean geometry and hyperbolic geometry. Then we will introduce parallel postulates near the end of this chapter. Further, we will begin by introducing several analytic models to illustrate and develop better understanding of the axioms and concepts. Also, we will restrict our study to plane geometry and forgo the axioms for space. We will use the plane geometry axioms from the SMSG axiom set.

This chapter assumes the reader has had at least a high school Euclidean geometry course that included proofs.

Use prepared software from Appendix B of the Course Title Page Prepared Geometer's Sketchpad Sketches, and *NonEuclid* java program for constructions in hyperbolic geometry http://cs.unm.edu/~joel/NonEuclid/NonEuclid.html.

Before we begin a formal study of neutral geometry leading to Euclidean and hyperbolic geometry, we emphasize the importance of the axioms and not making additional assumptions based on diagrams.

Exercise 2.1. Identify the error or errors in the proof that all triangles are isosceles.

Exercise 2.2. Identify the error or errors in the proof of the Rusty Compass Theorem.

O King, for traveling over the country, there are royal roads and roads for common citizens; but in geometry there is one road for all.
—Menaechmus (350 B.C.)