2.2 Incidence Axioms

Geometry enlightens the intellect and sets one's mind right.
—Ibn Khaldun (1332–1406)

Why do surveyors, photographers, and artists use tripods? How is the location of a football that has been punted out of bounds determined? The axioms for Euclidean geometry are such that the mathematics matches our real-world expectations. What axioms or theorems justify the solutions to the two questions?

The SMSG incidence axioms are Postulates 1 and 5–8; however, since we are only concerned with plane geometry, the only axioms that apply to our study of a neutral geometry are Postulates 1, 5(a), and 6.

Postulate 1. (*Line Uniqueness*) Given any two distinct points there is exactly one line that contains them.

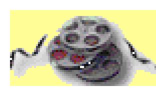
Postulate 5(a). (*Existence of Points*) Every plane contains at least three noncollinear points.

Postulate 6. (*Points on a Line Lie in a Plane*) If two points lie in a plane, then the line containing these points lies in the same plane.

Definition. A set of points S is *collinear* if there is a line l such that S is a subset of l. S is *noncollinear* if S is not a collinear set. If $\{A, B, C\}$ is a collinear set, we say that the points A, B, and C are collinear.

Theorem 2.2. Two distinct lines intersect in at most one point.

Proposition 2.3. The Cartesian plane satisfies SMSG Postulates 1, 5(a), and 6.



Proof. To show Postulate 5(a) is satisfied consider the three points (0, 0), (1, 0), and (0, 1), which are points in the Cartesian plane. The three points are not on the same vertical line, since they do not have the same first coordinate. Suppose they are on a nonvertical line $l_{m,b}$, then $0 = m(0) + b$, $0 = m(1) + b$, and $1 = m(0) + b$, i.e. $b = 1$ and $b = 0$, which is a contradiction. Hence (0, 0), (1, 0), and (0, 1) are three distinct noncollinear points in the Cartesian plane, and SMSG Postulate 5 is satisfied.

Next, show SMSG Postulate 1 is satisfied. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be distinct points in the Cartesian plane. Then x_1 and x_2 are either equal or not equal. We first show the existence of a line containing P and Q.

Case 1. Assume $x_1 = x_2$. Set $a = x_1 = x_2$. Thus P and Q are both on line $l_a = \{(x, y) : x = a\}$.

Case 2. Assume x_1 and x_2 are not equal. We need to find m and b such that P and Q are on line $l_{m,b}$. *(First, do some scratch work to find an m and b that will work. Based on the scratch work, define an m and b, and then show they work.)* Set $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and $b = y_2 - \dfrac{y_2 - y_1}{x_2 - x_1} \cdot x_2$. We show that P and Q are on the line $l_{m,b} = \{(x, y) : y = mx + b\}$. For P,

$$mx_1 + b = \frac{y_2 - y_1}{x_2 - x_1} \cdot x_1 + y_2 - \frac{y_2 - y_1}{x_2 - x_1} \cdot x_2$$
$$= \frac{y_2 - y_1}{x_2 - x_1}(x_1 - x_2) + y_2$$
$$= -(y_2 - y_1) + y_2 = y_1.$$

Hence P is on line $l_{m,b}$. For Q, $mx_2 + b = \frac{y_2 - y_1}{x_2 - x_1} \cdot x_2 + y_2 - \frac{y_2 - y_1}{x_2 - x_1} \cdot x_2 = y_2$. Hence Q is on line $l_{m,b}$.

Therefore, P and Q are on line $l_{m,b}$. Thus by the two cases, there is at least one line that contains P and Q.

We need to show there is exactly one line that contains P and Q. Suppose P and Q belong to lines l and k. There are three possible cases (1) both lines are vertical; (2) both lines are nonvertical; or (3) one line is vertical, and the other line is nonvertical.

Case 1. Assume $l = l_a$ and $k = k_b$. Then $a = x_1 = x_2 = b$. Hence l and k are the same line.

Case 2. Assume $l = l_{m,b}$ and $k = k_{n,c}$. Then $y_1 = mx_1 + b$, $y_2 = mx_2 + b$, $y_1 = nx_1 + c$, and $y_2 = nx_2 + c$. Solve this system of four equations:

$$mx_1 + b = nx_1 + c \text{ and } mx_2 + b = nx_2 + c$$
$$(m - n)\, x_1 = c - b \text{ and } (m - n)\, x_2 = c - b$$
$$(m - n)x_1 = (m - n)x_2$$
$$(m - n)(x_1 - x_2) = 0.$$

Since for nonvertical lines x_1 and x_2 are not equal, we have that $m = n$. Thus $c - b = 0$, i.e. $c = b$. Hence $l = l_{m,b} = k_{n,c} = k$.

Case 3. Without loss of generality, assume $l = l_a$ and $k = k_{n,c}$. Since P and Q are on l, $a = x_1 = x_2$. That is, $P = (a, y_1)$ and $Q = (a, y_2)$. Since P and Q are on k, $y_1 = nx_1 + c = na + c = nx_2 + c = y_2$. Hence $P = (x_1, y_1) = (x_2, y_2) = Q$. But this contradicts that P and Q are distinct. Therefore, this case is not possible.

By the above three cases, there is exactly one line that contains P and Q. Since the two points were arbitrarily chosen, we have that the Cartesian plane satisfies SMSG Postulate 1.

From how lines are defined for a Cartesian plane and the above proof that Postulate 1 is satisfied, it immediately follows that the Cartesian plane satisfies SMSG Postulate 6.//

Corollary to Proposition 2.3. The Euclidean plane, Taxicab plane, and Max-distance plane satisfy SMSG Postulates 1, 5(a), and 6.

Exercise 2.11. Find the axioms from a high school geometry book that correspond to SMSG Postulates 1 and 5.

Exercise 2.12. Prove Theorem 2.2.

Exercise 2.13. Show the Missing Strip Plane satisfies (a) SMSG Postulate 1 and (b) SMSG Postulate 5 (a).

Exercise 2.14. Show the Poincaré Half-plane satisfies (a) SMSG Postulate 1 and (b) SMSG Postulate 5 (a).

Exercise 2.15. Does a Discrete plane satisfy (a) SMSG Postulate 1? Justify. (b) SMSG Postulate 5(a)? Justify.

Exercise 2.16. Do the Riemann Sphere and Modified Riemann Sphere satisfy (a) SMSG Postulate 1? Justify. (b) SMSG Postulate 5(a)? Justify.