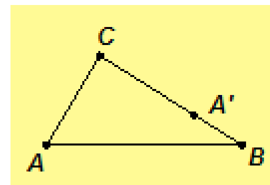
Triangle Inequality

Proof is the idol before whom the pure mathematician tortures himself.
—Sir Arthur Eddington (1882–1944)

On this page, we prove the Triangle Inequality based on neutral geometry results from Chapter 2.

Lemma. In a neutral geometry, if one angle is greater in measure than another angle of a triangle, then the opposite side of the greater angle is longer than the opposite side of the lessor angle. Furthermore, the longest side of a triangle is opposite the angle of greatest measure.

Proof. Given ΔABC. Assume $m(\angle BAC) > m(\angle CBA)$. Let A' be on ray $\overrightarrow{CB}$ such that $\overline{AC} \cong \overline{A'C}$. Then $\angle CAA' \cong \angle CA'A$. One of the following three is true: B is between C and A'; $A' = B$; or A' is between C and B.



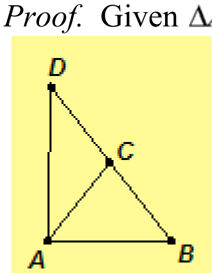
Case 1. Assume B is between C and A'. Then $B \in \text{int}(\angle A'AC)$ and by the Exterior Angle Theorem $m(\angle CBA) > m(\angle CA'A)$. Thus $m(\angle A'AC) = m(\angle A'AB) + m(\angle BAC) > m(\angle BAC)$. Hence $m(\angle CBA) > m(\angle CA'A) = m(\angle CAA') > m(\angle BAC)$, which contradicts the initial assumption. Hence B is not between C and A'.

Case 2. Assume $A' = B$. Then $\overline{AC} \cong \overline{BC}$. Hence $\angle BAC \cong \angle CBA$, which contradicts the initial assumption.

By Cases 1 and 2, we must have A' between C and B. Hence, $BC = BA' + A'C > A'C = AC$. The cases for the other pairs of sides may be proved similarly. Therefore, the longest side of a triangle is opposite the angle of greatest measure.//

Triangle Inequality. In a neutral geometry, the length of one side of a triangle is strictly less than the sum of the lengths of the other two sides.

Proof. Given ΔABC. We prove one case here; the other cases are similar. Let D be on ray $\overrightarrow{BC}$ such that



C is between B and D and $\overline{AC} \cong \overline{DC}$. Since ΔACD is an isosceles triangle, $\angle CAD \cong \angle CDA$. Since C is between B and D and $C \in \text{int}(\angle BAD)$, $m(\angle BAD) = m(\angle BAC) + m(\angle CAD)$, by the Angle Addition Postulate. Hence $m(\angle BAD) > m(\angle CAD) = m(\angle CDA) = m(\angle BDA)$. By the Lemma, $BD > BA$. Therefore, since C is between B and D, $AC + CB = DC + CB = DB > BA$. Since the proofs of the other two cases are similar, the length of one side of a triangle is strictly less than the sum of the lengths of the other two sides.//