2.6.2 Saccheri Quadrilaterals

The value of non-Euclidean geometry lies in its ability to liberate us from preconceived ideas in preparation for the time when exploration of physical laws might demand some geometry other than the Euclidean.
—Georg Friedrich Bernhard Riemann (1826–1866)

Euclid's Fifth Postulate. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Girolamo Saccheri (1667–1733), an Italian Jesuit priest and mathematician, attempted to prove Euclid's Fifth Postulate from the other axioms by the use of a *reductio ad absurdum* argument by assuming the negation of the Fifth Postulate. In 1733, shortly before his death, he published a book entitled *Euclides ab omni nævo vindicatus* (*Euclid Freed of Every Flaw/Blemish*), which represents some of the earliest work in non-Euclidean geometry. He believed he had demonstrated the result, but there were errors in his work where he had made additional assumptions based on preconceptions. Due to Saccheri's knowledge of logic and care in writing proofs, some believe that Saccheri knew of the errors. But left the errors in the work so that the Catholic church would not prevent publication since the results would have been in conflict with church doctrine of the time. The book remained obscure until rediscovered by the Italian mathematician Eugenio Beltrami (1835–1900).

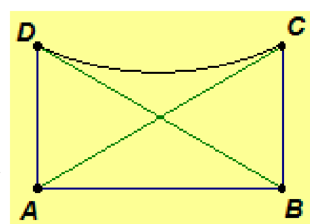
Saccheri considered a certain type of quadrilateral, called a *Saccheri quadrilateral,* as a basis for the beginning of his work in attempting to demonstrate the Fifth Postulate. We consider Saccheri quadrilaterals here to show just how close to the Euclidean concept of parallel lines one can arrive without assuming the Fifth Postulate. Since the existence of a rectangle is equivalent to Euclid's Fifth Postulate, the idea is to show the existence of a rectangle. Note the common properties of the Saccheri quadrilateral and a rectangle. In Euclidean geometry, a Saccheri quadrilateral is a rectangle.

Definition. A *Saccheri quadrilateral* is a quadrilateral *ABCD* where $\angle BAD$ and $\angle ABC$ are right angles and $\overline{AD} \cong \overline{BC}$. Segment $\overline{AB}$ is called the *base,* and segment $\overline{CD}$ is called the *summit*.

A *parallelogram* is a quadrilateral in which both pairs of opposite sides are parallel.

A *rectangle* is a quadrilateral with four right angles.

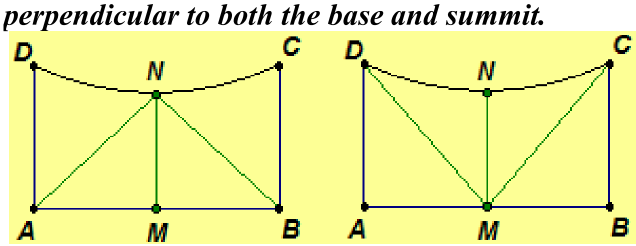
Theorem 2.16. The diagonals of a Saccheri quadrilateral are congruent.



Theorem 2.17. The summit angles of a Saccheri quadrilateral are congruent.

Proof. Given Saccheri quadrilateral *ABCD* with right angles $\angle BAD$ and $\angle ABC$, and $\overline{AD} \cong \overline{BC}$. By Theorem 2.16, $\overline{AC} \cong \overline{BD}$. Since $\overline{AD} \cong \overline{BC}$, $\overline{AC} \cong \overline{BD}$, and $\overline{CD} \cong \overline{DC}$, we have $\Delta ACD \cong \Delta BDC$. Hence $\angle ADC \cong \angle BCD$. Therefore, the summit angles of a Saccheri quadrilateral are congruent.//

Theorem 2.18. The segment joining the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to both the base and summit.

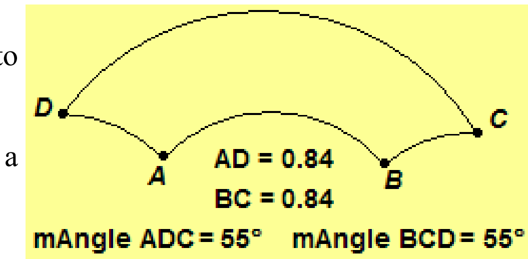


Proof. Given Saccheri quadrilateral *ABCD* with right angles $\angle BAD$ and $\angle ABC$, and $\overline{AD} \cong \overline{BC}$. Further, assume *M* and *N* are the midpoints of segment *AB* and segment *CD*, respectively. Thus *A-M-B*, *C-N-D*, $\overline{AM} \cong \overline{MB}$, and $\overline{DN} \cong \overline{NC}$. Since the summit angles are congruent, $\angle ADN = \angle ADC \cong \angle BCD = \angle BCN$. Since $\overline{AD} \cong \overline{BC}$,

$\angle ADN \cong \angle BCN$, and $\overline{DN} \cong \overline{NC}$, we have $\Delta ADN \cong \Delta BCN$. Hence $\overline{AN} \cong \overline{BN}$. Since $\overline{AM} \cong \overline{MB}$, $\overline{AN} \cong \overline{BN}$, and $\overline{MN} \cong \overline{MN}$, we have $\Delta AMN \cong \Delta BMN$. Hence $\angle AMN \cong \angle BMN$. Since *A*-*M*-*B*, $\angle AMN$ and $\angle BMN$ are a linear pair. Since a linear pair of congruent angles are right angles, $\angle AMN$ and $\angle BMN$ are right angles. Hence, by the definition of perpendicular lines, line *AB* is perpendicular to line *MN*. A similar procedure may be used to prove line *CD* is perpendicular to line *MN*. Therefore, segment *MN* is perpendicular to both segment *AB* and segment *CD*.//

Theorem 2.19. The summit and base of a Saccheri quadrilateral are parallel.

Theorem 2.20. A Saccheri quadrilateral is a parallelogram.



It can be proven that the existence of a rectangle is equivalent to Euclid's Fifth Postulate. Notice, Theorems 2.16 –2.20, the properties that a Saccheri quadrilateral and a rectangle have in common. But, the Saccheri quadrilateral is not a rectangle without a Euclidean parallel postulate. For an example of a Saccheri quadrilateral that is not a rectangle, consider the Saccheri quadrilateral in the Poincaré Half-plane on the right. The summit angles at *C* and *D* are not right angles, since their value is less than 90. The Poincaré Half-plane is a model of a hyperbolic geometry in which it can be shown that the summit angles of a Saccheri quadrilateral measure less than 90.

Exercise 2.62. Prove Theorem 2.16.

Exercise 2.63. Prove Theorem 2.19.

Exercise 2.64. Prove Theorem 2.20.