



## 2.7.3 Elliptic Parallel Postulate

*The postulate on parallels...was in antiquity the final solution of a problem that must have preoccupied Greek mathematics for a long period before Euclid.*

—  [Hans Freudenthal \(1905–1990\)](#)

**Elliptic Parallel Postulate.** Any two lines intersect in at least one point.


An important note is how elliptic geometry differs in an important way from either Euclidean geometry or hyperbolic geometry. Whereas, Euclidean geometry and hyperbolic geometry are neutral geometries with the addition of a parallel postulate, elliptic geometry cannot be a neutral geometry due to [Theorem 2.14](#), which stated that parallel lines exist in a neutral geometry. Hence, the Elliptic Parallel Postulate is [inconsistent](#) with the axioms of a neutral geometry. Elliptic geometry requires a different set of axioms for the axiomatic system to be consistent and contain an elliptic parallel postulate.

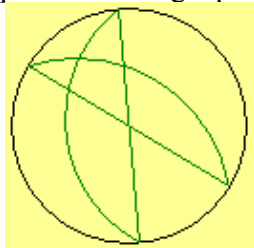
 [Georg Friedrich Bernhard Riemann](#) (1826–1866) was the first to recognize that the geometry on the surface of a sphere, spherical geometry, is a type of non-Euclidean geometry. This is the reason we name the spherical model for elliptic geometry after him, the [Riemann Sphere](#). (To help with the visualization of the concepts in this section, use a ball or a globe with rubber bands or string.) [Click here to download](#)  [Spherical Easel](#) a java exploration of the Riemann Sphere model. In a spherical model:

- Two lines, which are great circles, intersect in two points called *poles* or *antipodal points*.
- A line is unbounded, by this it is meant that a line has no endpoints. A great circle has no beginning and no end.
- A line has finite length.
- The concept of betweenness of points does not make sense. What does it mean for a point to be between two other points on a line (great circle)?


From these properties of a sphere, we see that in order to formulate a consistent axiomatic system, several of the axioms from a neutral geometry need to be dropped or modified, whether using either Hilbert's or Birkhoff's axioms. The incidence axiom that "*any two points determine a unique line,*" needs to be modified to read "*any two points determine at least one line.*" Hilbert's Axioms of Order (betweenness of points) may be replaced with axioms of separation that give the properties of how points of a line separate each other. (For a listing of separation axioms see *Euclidean and Non-Euclidean Geometries Development and History* by [Greenberg](#).) With these modifications made to the axiom system, the Elliptic Parallel Postulate may be added to form a consistent system. Often spherical geometry is called *double elliptic geometry*, since two distinct lines intersect in two points.

One problem with the spherical geometry model is that two lines intersect in more than one point.

 [Felix Klein](#) (1849–1925) modified the model by identifying each pair of antipodal points as a single point, see the [Modified Riemann Sphere](#). With this model, the axiom that any two points determine a unique line is satisfied. Often an elliptic geometry that satisfies this axiom is called a *single elliptic geometry*. Note that with this model, a line no longer separates the plane into distinct half-planes, due to the association of antipodal points as a single point.



Klein formulated another model for elliptic geometry through the use of a circle. The model is similar to the Poincaré Disk. Given a Euclidean circle, a point in the model is of two types: a point in the interior of the Euclidean circle or a point formed by the identification of two antipodal points which are the endpoints of a diameter of the Euclidean circle. The lines are of two types: diameters of the Euclidean circle or arcs of Euclidean circles that intersect the given Euclidean circle at the endpoints of diameters of the given circle. The model on the left illustrates four lines, two of each type. The model can be viewed as taking the Modified Riemann Sphere and flattening onto a

Euclidean plane. [Click here for a](#)  [javasketchpad construction that uses the Klein model](#).

*Exercise 2.75.* In the Riemann Sphere, what properties are true about all lines perpendicular to a given line?

*Exercise 2.76.* How does a Möbius strip relate to the Modified Riemann Sphere?

*Exercise 2.77.* Describe how it is possible to have a triangle with three right angles.

*Exercise 2.78.* Find an upper bound for the sum of the measures of the angles of a triangle in the Riemann Sphere.

*Exercise 2.79.* Use a ball to represent the Riemann Sphere, construct a Saccheri quadrilateral on the ball. Are the summit angles acute, right, or obtuse? Is the length of the summit more or less than the length of the base? (Remember the sides of the quadrilateral must be segments of great circles.)

*Non-Euclidean space is the false invention of demons, who gladly furnish the dark understanding of the non-Euclideans with false knowledge... The non-Euclideans, like the ancient sophists, seem unaware that their understandings have become obscured by the promptings of the evil spirits.*

—Matthew Ryan (1905)

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