3.2.1 Preliminary Definitions and Assumptions

Knowing is not enough; we must apply.
Willing is not enough; we must do.
—Johann von Goethe (1749–1842)

Definition. A *mapping* (or function f from A to B) of a set A into a set B is a rule that pairs each element of A with exactly one element of a subset of B. The set A is called the *domain*, and the set of all elements of B (a subset of B) that are paired with an element from A is called the *range*.

Definition. A mapping f from A to B is *onto* B if for any b in B there is at least one a in A such that $f(a) = b$.

Definition. A mapping f from A to B is *one-to-one* if each element of the range of f is the image of exactly one element from A.

Definition. A *transformation* is a one-to-one mapping of a set A onto a set B.

Definition. A *transformation of a plane* is a transformation that maps points of the plane onto points in the plane.

Definition. A nonempty set G is said to form a *group under a binary operation*, *, if it satisfies the following conditions:

i. If A and B are in G, then $A*B$ is in G. (The set is *closed* under the operation, *closure*.)
ii. There exists an element I in G such that for every element A in G, $I*A = A*I = A$. (The set has an *identity*.)
iii. For every element A in G, there is an element B in G such that $A*B = B*A = I$, denoted A^{-1}. (Every element has an *inverse*.)
iv. If A, B, and C are in G, then $(A*B)*C = A*(B*C)$. (*associativity*)

Theorem 3.0. The set of transformations of a plane is a group under composition.

Proof. The result follows from the following:
The composition of two transformations of a plane is a transformation (Exercise 3.4).
The inverse of a transformation is a transformation (Exercise 3.5).
The identity function is a transformation and composition of functions is associative (Exercise 3.5).//

Exercise 3.2. Which of the following mappings are transformations? Justify.

a. $f:\mathbb{R}\to\mathbb{R}$ such that $f(x)=\frac{x-3}{2}$.

b. $f:\mathbb{R}\to\mathbb{R}$ such that $f(x)=x^2$.

c. $f:\mathbb{R}^2\to\mathbb{R}^2$ such that $f(x,y)=(x-2,y+1)$.

d. $f:\mathbb{R}^2\to\mathbb{R}^2$ such that $f(x,y)=(2x,3y)$.

e. Let P be a point in a plane S. Define $f:S\to S$ by $f(P) = P$ and for any point $Q \neq P$, $f(Q)$ is the midpoint of $\overline{PQ}$.

Exercise 3.3. Let $f:\mathbb{R}^2 \to \mathbb{R}^2$ and $g:\mathbb{R}^2 \to \mathbb{R}^2$ be transformations defined respectively by $f(x,y) = (x - 4, y + 1)$ and $g(x,y) = (x + 2, y + 3)$.

a. Find the composition $f \circ g$.

b. Find the composition $g \circ f$.

c. Find the inverse of f, f^{-1}.

d. Find the inverse of g, g^{-1}.

Exercise 3.4. Prove the composition of two transformations of a plane is a transformation of the plane.

Exercise 3.5. (a) Prove the identity function is a transformation. (b) Prove the inverse of a transformation of a plane is a transformation of the plane. (c) Prove the composition of functions is associative.