3.2.3 Affine Transformation of the Euclidean Plane

A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.
—Godfrey Harold Hardy (1877–1947)

What is the form of a transformation matrix for the analytic model of the Euclidean plane? We investigate this question. Let $A = [a_{ij}]$ be a transformation matrix for the Euclidean plane and $(x, y, 1)$ be any point in the Euclidean plane. Then

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y + a_{13} \\ a_{21}x + a_{22}y + a_{23} \\ a_{31}x + a_{32}y + a_{33} \end{bmatrix}.$$

Since the last matrix must be the matrix of a point in the Euclidean plane, we must have $a_{31}x + a_{32}y + a_{33} = 1$ for every point $(x, y, 1)$ in the Euclidean plane. In particular, the point $(0, 0, 1)$ must satisfy the equation. Hence, $a_{33} = 1$. Further, the points $(0, 1, 1)$ and $(1, 0, 1)$ satisfy the equation and imply $a_{32} = 0$ and $a_{31} = 0$, respectively. Therefore, the transformation matrix must have the form

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix},$$

which motivates the following definition.

Definition. An *affine transformation of the Euclidean plane, T,* is a mapping that maps each point X of the Euclidean plane to a point $T(X)$ of the Euclidean plane defined by $T(X) = AX$ where $\det(A)$ is nonzero and

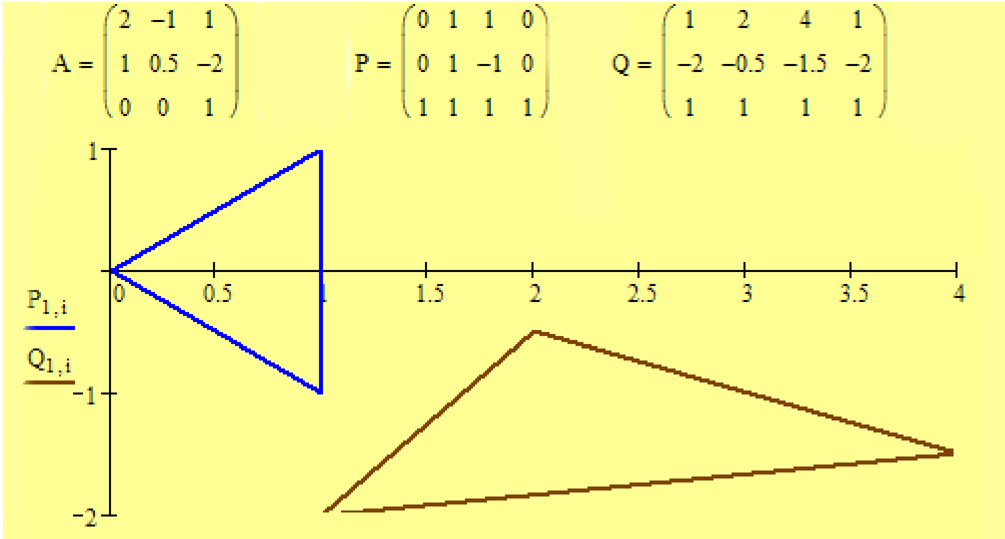
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$$ where each a_{ij} is a real number.

Exercise 3.19. Prove that every affine transformation of the Euclidean plane has an inverse that is an affine transformation of the Euclidean plane. *(Hint. Write the inverse by using the adjoint. Refer to a linear algebra text.)*

Proposition 3.3. An affine transformation of the Euclidean plane is a transformation of the Euclidean plane.

Exercise 3.20. Prove Proposition 3.3.

Click here to see an animation of a sequence of affine transformations.



Matrix *A* is the transformation matrix.
Matrix *P* is the matrix of the three endpoints of the original triangle.
Matrix *Q* = *AP* is the matrix of the transformed endpoints.

Proposition 3.4. The set of affine transformations of the Euclidean plane form a group under matrix multiplication.

Proof. Since the identity matrix is clearly a matrix of an affine transformation of the Euclidean plane and the product of matrices is associative, we need only show closure and that every transformation has an inverse.

Let A and B be the matrices of affine transformations of the Euclidean plane. Since det(A) and det(B) are both nonzero, we have that det(AB) = det(A) · det(B) is not zero. Also,

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} & a_{11}b_{13}+a_{12}b_{23}+a_{13} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} & a_{21}b_{13}+a_{22}b_{23}+a_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

is a matrix of an affine transformation of the Euclidean plane. *(The last row of the matrix is 0 , 0, 1.)* Hence closure holds.

Complete the proof by showing the inverse property.//

Exercise 3.21. Given three points $P(0, 0, 1)$, $Q(1, 0, 1)$, and $R(2, 1, 1)$, and an affine transformation T.
(a) Find the points $P' = T(P)$, $Q' = T(Q)$, and $R' = T(R)$ where the matrix of the transformation is $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$. (b) Sketch triangle PQR and triangle $P'Q'R'$. (c) Describe how the transformation moved and changed the triangle PQR.

Exercise 3.22. Find the matrix of an affine transformation that maps $P(0, 0, 1)$ to $P'(0, 2, 1)$, $Q(1, 0, 1)$ to $Q'(2, 1, 1)$, and $R(2, 3, 1)$ to $R'(7, 9, 1)$.

Exercise 3.23. Show the group of affine transformations of the Euclidean plane is not commutative.